EXACT FORMULA FOR THE SECOND-ORDER TANGENT SET OF THE SECOND-ORDER CONE COMPLEMENTARITY SET[*]

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Abstract. The second-order tangent set is an important concept in describing the curvature of the set involved. Due to the existence of the complementarity condition, the second-order cone (SOC) complementarity set is a nonconvex set. Moreover, unlike the vector complementarity set, the SOC complementarity set is not even the union of finitely many polyhedral convex sets. Despite these difficulties, we succeed in showing that like the vector complementarity set, the SOC complementarity set is second-order directionally differentiable and an exact formula for the second-order tangent set of the SOC complementarity set can be given. We derive these results by establishing the relationship between the second-order tangent set of the SOC complementarity set and the second-order directional derivative of the projection operator over the SOC, and calculating the second-order directional derivative of the projection operator over the SOC. As an application, we derive second-order necessary optimality conditions for the mathematical program with SOC complementarity constraints.

Key words. projection operator, second-order directional derivatives, second-order tangent sets, second-order cone complementarity sets, second-order necessary optimality conditions, mathematical program with second-order cone complementarity constraints

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1. Introduction. In optimization, an important issue is how to approximate the feasible region using derivatives of the function and the tangent cone of the set involved. Such needs arise in optimality conditions, constraint qualifications, and stability analysis when the problem data are perturbed. In the same way that second-order derivatives provide quadratic approximations whereas first-order derivatives only provide linear approximation to a given function, second-order tangent sets provide better approximation than tangent cones to a set at a point, in particular when the given set is not a polyhedral set or the union of finitely many polyhedral sets. As a result, the second-order tangent sets have been used successfully in second-order optimality conditions, stability analysis, and metric subregularity (see, e.g., [2, 3, 4, 7, 8, 10, 13, 16] and references therein). More recently, Gfrerer and Mordukhovich [11] use the second-order tangent set to give an estimate of the upper curvature of a set, which is used to study the Robinson stability of parametric constraint systems.

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In optimization, one often has to deal with a feasible region in the form $C :=$ $\{x \mid F(x) \in \Theta\}$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is a second-order continuously differentiable mapping and Θ is a closed set in $\mathbb{R}^m$. By [18, Proposition 13.13], under a constraint qualification, the second-order tangent set of the feasible region C can be characterized as

$$(1) \quad \left. \begin{array}{l} d \in T_C(x), \\ w \in T_C^2(x,d) \end{array} \right\} \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} \nabla F(x)d \in T_\Theta(F(x)), \\ \nabla F(x)w + d^T \nabla^2 F(x)d \in T_\Theta^2(F(x); \nabla F(x)d), \end{array} \right.$$

where T_C, T_C^2 denote the tangent cone and the second-order tangent set, respectively (see Definition 2.1). In the case when $\Theta = \mathbb{R}_-^{m_1} \times \{0\}^{m_2}$, $m_1 + m_2 = m$, the system is described by inequality and equality constraints. In this case, since the set Θ is polyhedral, the second-order tangent set of $\mathbb{R}_-^{m_1} \times \{0\}^{m_2}$ is a polyhedral set, and hence the second-order tangent set of the feasible region is a system of equalities and inequalities involving the second-order derivatives of the constraint mapping F (see, e.g., Bonnans and Shapiro [4, equation (3.81)]), provided that a constraint qualification holds. In recent years, the second-order cone programming (SOCP) has attracted much attention due to a broad range of applications in fields from engineering, control, and finance to robust optimization and combinatorial optimization (see, e.g., [1] for an introduction to the theory and its applications).

Consider the second-order cone defined as

$$\mathcal{K} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid \|x_2\| \leq x_1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Bonnans and Ramírez gave the characterization for the second-order tangent set [3, Lemma 27], and used it to formulate second-order necessary and sufficient optimality conditions for nonlinear SOCPs. Since the second-order cone is not polyhedral, the second-order tangent set is not polyhedral [3].

In recent years, there has been more and more research on the second-order cone (SOC) complementarity system defined as

$$\mathcal{K} \ni G(z) \perp H(z) \in \mathcal{K},$$

where $u \perp v$ means the vectors u and v are perpendicular, $G(z), H(z) : \mathbb{R}^n \to \mathbb{R}^m$. One of the sources of the SOC complementarity system is the Karush–Kuhn–Tucker (KKT) optimality condition for second-order cone programming (see, e.g., [1, 5]), and the other is the equilibrium system for a Nash game where the constraints involve second-order cones (see, e.g., [14]). We call the closed cone

$$\Omega := \{(x, y) \in \mathbb{R}^{2m} \mid \mathcal{K} \ni x \perp y \in \mathcal{K}\}$$

the SOC complementarity set (or the complementarity set associated with the second-order cone; c.f. [15]). Using the SOC complementarity set, the SOC complementarity system can be reformulated as $(G(z), H(z)) \in \Omega$. Due to the existence of the complementarity condition, the SOC complementarity set is a nonconvex set. Moreover, due to the nonpolyhedral structure of the second-order cone $\mathcal{K}$, the SOC complementarity set is also nonpolyhedral. Hence the SOC complementarity set is a difficult object to study in the variational analysis.

The main goal of this paper is to provide a precise formula for the second-order tangent set to the SOC complementarity set Ω. The projection operator over the second-order cone $\Pi_\mathcal{K}(x) := \arg\min_{x' \in \mathcal{K}} \|x' - x\|$ is one of our main tools in the

subsequent analysis. It is well known that the metric projection operator $\Pi_\mathcal{K}(x)$ provides an alternative characterization of the SOC complementarity set:

$$(2) \qquad\qquad (x, y) \in \Omega \Longleftrightarrow \Pi_\mathcal{K}(x - y) = x.$$

The projection operator $\Pi_\mathcal{K}(x)$ is known to be first-order directionally differentiable (see, e.g., [17, Lemma 2]) and the connection between its tangent cone and its directional derivative has been given (see [15, 21]): for any $(x, y) \in \Omega$,

$$(3) \qquad\qquad (d, w) \in T_\Omega(x, y) \Longleftrightarrow \Pi'_\mathcal{K}(x - y; d - w) = d.$$

Using this connection, it has been shown that the SOC complementarity set Ω is geometrically derivable and the exact formula for its tangent cone is given; see, e.g., [21, Theorem 5.1]. Moreover, the coderivative of the projection operator $\Pi_\mathcal{K}$ allows us to characterize the various normal cones as in [20, Proposition 2.1] and show that the SOC complementarity set is not only geometrically derivable but also directionally regular [21, Theorem 6.1]. So far, by using the first-order variational analysis, it has been revealed that although the SOC complementarity set is neither a convex set nor the union of finitely many polyhedral convex sets, it enjoys certain nice properties that a convex set or the union of finitely many polyhedral convex sets have. In this paper, we continue to investigate the second-order variational properties of the SOC complementarity cone. Our main contributions are as follows.

- We derive the exact formula for the second-order directional derivative of the projection operator over the second-order cone. We further establish the connection between the second-order tangent set and the second-order directional derivative of the projection operator: for any $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$,

$$(4) \qquad (p, q) \in T^2_\Omega((x, y); (d, w)) \Longleftrightarrow \Pi''_\mathcal{K}(x - y; d - w, p - q) = p.$$

- We show that the SOC complementarity set is second-order directionally differentiable (see Definition 2.2). Note that this nice property is not even enjoyed by a convex set (see [4, Example 3.31]).
- Using the characterization (4) and the precise formula for the second-order directional derivative of the projection operation over the second-order cone, we derive the exact formula for the second-order tangent set of the SOC complementarity set. Compared with the usual vector complementarity set, our research shows that the task of establishing the formula of the second-order tangent set to the second-order cone complementarity set, which has nonpolyhedral and nonconvex structure, is not trivial.
- Based on the exact formula of the second-order tangent set of Ω, we develop the second-order optimality conditions for the mathematical program with second-order cone complementarity constraints (SOCMPCC).

We organize our paper as follows. Section 2 contains the preliminaries. In section 3, we calculate the second-order directional derivative of the projection operator over the second-order cone. Section 4 is devoted to the exact formula of the second-order tangent set to the SOC complementarity set. The second-order optimality conditions of SOCMPCC are discussed in section 5.

2. Preliminaries. In this section, we clarify the notation and recall some background material. First, we denote by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the set of nonnegative scalars and positive scalars, respectively, i.e., $\mathbb{R}_+ := \{\alpha \mid \alpha \geq 0\}$ and $\mathbb{R}_{++} := \{\alpha \mid \alpha > 0\}$. For a

set C, denote by $\operatorname{int}C$, $\operatorname{cl}C$, $\operatorname{bd}C$, $\operatorname{co}C$, C^c its interior, closure, boundary, convex hull, and complement, respectively. For a closed set $C \subseteq \mathbb{R}^n$, let C° and $\sigma(\cdot|C)$ stand for the polar cone and the support function of C, respectively, i.e.,

$$C^\circ := \{v \mid \langle v, w \rangle \leq 0 \ \forall w \in C\}$$

and $\sigma(z|C) := \sup\{\langle z, x \rangle \mid x \in C\}$ for $z \in \mathbb{R}^n$. Denote by $\operatorname{lin}C$ the largest subspace L such that $C + L \subseteq C$. For a vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we denote by x° the polar set of the set $\{x\}$ and by $\hat{x} := (x_1, -x_2)$ the reflection of vector x on the x_1 axis. For a nonzero vector x, we define $\bar{x} := x/\|x\|$. Let $o(\lambda) : \mathbb{R}_+ \to \mathbb{R}^m$ stand for a mapping with the property that $o(\lambda)/\lambda \to 0$ when $\lambda \downarrow 0$. For a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ and vectors $x, d \in \mathbb{R}^n$, we denote by $\nabla F(x) \in \mathbb{R}^{m \times n}$ the Jacobian of F at x, by $\nabla^2 F(x)$ the second-order derivative of F at x, and by $\nabla^2 F(x)(d, d)$ the quadratic form corresponding to $\nabla^2 F(x)$. The directional derivative of F at x in direction d is defined as

$$F'(x; d) := \lim_{t \downarrow 0} \frac{F(x + td) - F(x)}{t},$$

provided that the above limit exists. If F is directionally differentiable at x in direction d, its parabolic second-order directional derivative is defined as

$$F''(x; d, w) := \lim_{t \downarrow 0} \frac{F(x + td + \frac{1}{2}t^2 w) - F(x) - tF'(x; d)}{\frac{1}{2}t^2},$$

provided that the above limit exists. Moreover if the limit

$$F''(x; d, w) = \lim_{\substack{t \downarrow 0, \\ w' \to w}} \frac{F(x + td + \frac{1}{2}t^2 w') - F(x) - tF'(x; d)}{\frac{1}{2}t^2}$$

exists, then F is said to be parabolical second-order directionally differentiable at x in the direction d in the sense of Hadamard. In general, the concept of parabolical second-order directional differentiability in the Hadamard sense is stronger than that of parabolical second-order directional differentiability. However, when F is locally Lipschitz at x, these two concepts coincide. It is known that if F is parabolical second-order directional differentiable in the Hadamard sense at x along d, w, then

$$F\left(x + td + \frac{1}{2}t^2 w + o(t^2)\right) = F(x) + tF'(x; d) + \frac{1}{2}t^2 F''(x; d, w) + o(t^2).$$

DEFINITION 2.1 (tangent cones). *Let $S \subseteq \mathbb{R}^m$ and $x \in S$. The regular/Clarke, inner, and (Bouligand–Severi) tangent/contingent cone to S at x are defined respectively as*

$$\widehat{T}_S(x) := \liminf_{\substack{x' \xrightarrow{S} x, \\ t \downarrow 0}} \frac{S - x'}{t} = \left\{d \in \mathbb{R}^m \ \middle| \ \forall t_k \downarrow 0, \ x_k \xrightarrow{S} x, \ \exists d_k \to d \text{ with } x_k + t_k d_k \in S\right\},$$

$$T_S^i(x) := \liminf_{t \downarrow 0} \frac{S - x}{t} = \left\{d \in \mathbb{R}^m \ \middle| \ \forall t_k \downarrow 0, \ \exists d_k \to d \text{ with } x + t_k d_k \in S\right\},$$

$$T_S(x) := \limsup_{t \downarrow 0} \frac{S - x}{t} = \left\{d \in \mathbb{R}^m \ \middle| \ \exists t_k \downarrow 0, \ d_k \to d \text{ with } x + t_k d_k \in S\right\}.$$

The inner and outer second-order tangent sets to S at x in direction d are defined respectively as

$$T_S^{i,2}(x;d) := \left\{ w \in \mathbb{R}^m \ \middle| \ \text{dist}\left(x + td + \frac{1}{2}t^2 w, S\right) = o(t^2), \ t \geq 0 \right\},$$

$$T_S^2(x;d) := \left\{ w \in \mathbb{R}^m \ \middle| \ \exists \ t_n \downarrow 0 \text{ such that } \text{dist}\left(x + t_n d + \frac{1}{2}t_n^2 w, S\right) = o(t_n^2) \right\}.$$

While for a nonconvex set S, the contingent cone $T_S(x)$ may be nonconvex, it is known that the regular/Clarke tangent cone $\widehat{T}_S(x)$ is always closed and convex. By definition, since the distance function of a convex set is convex, it is easy to see that the inner second-order tangent set is always convex when the set S is convex. On the other hand, the outer second-order tangent set may be nonconvex even when the set S is convex (see [4, Example 3.35]). Note that $T_S^{i,2}(x;d) \subseteq T_S^2(x;d)$ and the outer second-order tangent set $T_S^2(x;d)$ need not be a cone (it may be empty; see, for instance, an example in [18, Page 592]). If $T_S^{i,2}(x;d) = T_S^2(x;d)$, we simply call $T_S^2(x;d)$ the second-order tangent set to S at x in direction d.

DEFINITION 2.2 (see [4, Definition 3.32]). *A set S is said to be second-order directionally differentiable at $x \in S$ in a direction $d \in T_S(x)$ if $T_S^i(x) = T_S(x)$ and $T_S^{i,2}(x;d) = T_S^2(x;d)$.*

DEFINITION 2.3 (normal cones). *Let $S \subseteq \mathbb{R}^m$ and $x \in S$. The regular/Fréchet, limiting/Mordukhovich, and Clarke normal cone of S at x are defined respectively as*

$$\widehat{N}_S(x) := \left\{ v \in \mathbb{R}^m \ \middle| \ \langle v, x' - x \rangle \leq o(\|x' - x\|) \ \forall x' \in S \right\},$$

$$N_S(x) := \limsup_{x' \xrightarrow{S} x} \widehat{N}_S(x') = \left\{ \lim_{k \to \infty} v_k \ \middle| \ v_k \in \widehat{N}_S(x_k), \ x_k \xrightarrow{S} x \right\},$$

$$N_S^c(x) := \text{clco} N_S(x).$$

LEMMA 2.4 (tangent-normal polarity (see [18, Theorem 6.28], [6])). *For a closed set $S \subseteq \mathbb{R}^m$ and $x \in S$, $\widehat{T}_S(x) = (N_S(x))^\circ = (N_S^c(x))^\circ$, $\widehat{N}_S(x) = (T_S(x))^\circ$, $(\widehat{T}_S(x))^\circ = N_S^c(x)$.*

We recall some known results concerning the second-order cone $\mathcal{K}$ in $\mathbb{R}^m$. The topological interior and the boundary of $\mathcal{K}$ are

$$\text{int}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid x_1 > \|x_2\|\}$$

and

$$\text{bd}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid x_1 = \|x_2\|\},$$

respectively. Similar to the eigenvalue decomposition of a matrix, for any given vector $x := (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, x can be decomposed as (see, e.g., [9])

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i(x) := x_1 + (-1)^i \|x_2\| \quad \text{and} \quad u_x^{(i)} := \begin{cases} \frac{1}{2}(1, (-1)^i \bar{x}_2) & \text{if } x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if } x_2 = 0, \end{cases}$$

with w being a fixed unit vector in $\mathbb{R}^{m-1}$.

LEMMA 2.5 (see [3]). *For any $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, the following equivalence holds:*

$$x^T y = 0 \Longleftrightarrow y = k\hat{x} \text{ with } k = y_1/x_1 > 0 \Longleftrightarrow y = k\hat{x} \text{ with } k \in \mathbb{R}_{++}.$$

For a given real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define the SOC function $f^{\mathrm{soc}} :$ $\mathbb{R}^m \to \mathbb{R}^m$ as

$$(5) \qquad f^{\mathrm{soc}}(z) := f\left(\lambda_1(z)\right) u_z^{(1)} + f\left(\lambda_2(z)\right) u_z^{(2)}.$$

For $z \in \mathbb{R}^m$, let $\Pi_{\mathcal{K}}(z)$ be the metric projection of z onto $\mathcal{K}$. Then by [9], it can be calculated as

$$(6) \qquad \Pi_{\mathcal{K}}(z) = \lambda_1(z)_+ u_z^{(1)} + \lambda_2(z)_+ u_z^{(2)},$$

where $\alpha_+ := \max\{\alpha, 0\}$ is the nonnegative part of the number $\alpha \in \mathbb{R}$. Hence the projection operator $\Pi_{\mathcal{K}}(\cdot)$ is an SOC function corresponding to the plus function $f(\alpha) := \alpha_+$.

3. Second-order directional derivative of the projection operator over the second-order cone. As commented in the introduction, there exists a close relationship between the second-order tangent set of the SOC complementarity set and the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$; see (4). Therefore, to obtain the exact formula of the second-order tangent set, we need to calculate the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$. This task is done in this section, which is of independent interest. For convenience of notation, we sometime use $\Phi(x)$ instead of $\bar{x}$ to stand for $x/\|x\|$ as $x \neq 0$. It is easy to verify (see, e.g., [23, Theorem 3.1]) that Φ is second-order continuously differentiable at $x \neq 0$ with

$$\nabla\Phi(x) = (I - \bar{x}\bar{x}^T)/\|x\|,$$
$$\nabla^2\Phi(x)(w, w) = -2\frac{\bar{x}^T w}{\|x\|^2} w + w^T \left(\frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}\right) wx$$
$$= -2\frac{\bar{x}^T w}{\|x\|}\nabla\Phi(x)(w) - \frac{1}{\|x\|}w^T\nabla\Phi(x)w\bar{x},$$

where I is the identity matrix in $\mathbb{R}^{m \times m}$.

Since the second-order cone $\mathcal{K}$ is a special circular cone $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid \cos\theta\|x\| \leq x_1\}$$

with $\theta = 45°$, the SOC function f^{soc} is a special case of the circular cone function $f^{\mathcal{L}_\theta}$ studied in [23] with $\theta = 45°$. The following result follows from [23, Theorem 3.3] immediately.

LEMMA 3.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then, the SOC function f^{soc} is parabolic second-order directionally differentiable at x in the Hadamard sense if and only if f is parabolic second-order directionally differentiable at $\lambda_i(x)$ in the Hadamard sense for $i = 1, 2$. Moreover,*
  (i) *if $x_2 = 0$ and $d_2 = 0$, then*

$$\left(f^{\mathrm{soc}}\right)''(x; d, w) = f''\left(x_1; d_1, w_1 - \|w_2\|\right) u_w^{(1)} + f''\left(x_1; d_1, w_1 + \|w_2\|\right) u_w^{(2)};$$

(ii) *if $x_2 = 0$ and $d_2 \neq 0$, then*

$$
\begin{aligned}
(f^{\mathrm{soc}})''(x; d, w) = {}& f''\left(x_1; d_1 - \|d_2\|, w_1 - \bar{d}_2^T w_2\right) u_d^{(1)} \\
& + f''\left(x_1; d_1 + \|d_2\|, w_1 + \bar{d}_2^T w_2\right) u_d^{(2)} \\
& + \frac{1}{2}\left(f'(x_1; d_1 + \|d_2\|) - f'(x_1; d_1 - \|d_2\|)\right)\begin{pmatrix} 0 \\ \nabla\Phi(d_2)w_2 \end{pmatrix};
\end{aligned}
$$

(iii) *if $x_2 \neq 0$, then*

$$
\begin{aligned}
(f^{\mathrm{soc}})''&(x; d, w) \\
= {}& f''\left(x_1 - \|x_2\|; d_1 - \bar{x}_2^T d_2, w_1 - \left[\bar{x}_2^T w_2 + d_2^T \nabla\Phi(x_2)d_2\right]\right) u_x^{(1)} \\
& + f''\left(x_1 + \|x_2\|; d_1 + \bar{x}_2^T d_2, w_1 + \left[\bar{x}_2^T w_2 + d_2^T \nabla\Phi(x_2)d_2\right]\right) u_x^{(2)} \\
& + \left(f'(x_1 + \|x_2\|; d_1 + \bar{x}_2^T d_2) - f'(x_1 - \|x_2\|; d_1 - \bar{x}_2^T d_2)\right)\begin{pmatrix} 0 \\ \nabla\Phi(x_2)d_2 \end{pmatrix} \\
& + \frac{1}{2}\left(f(x_1 + \|x_2\|) - f(x_1 - \|x_2\|)\right)\begin{pmatrix} 0 \\ \nabla\Phi(x_2)w_2 + \nabla^2\Phi(x_2)(d_2, d_2) \end{pmatrix}.
\end{aligned}
$$

Since the projection operator $\Pi_{\mathcal{K}}(\cdot)$ is the SOC function corresponding to the plus function $f(\alpha) := \alpha_+$, we will need the second-order directional derivative of the plus function.

LEMMA 3.2 (see [22]). *Let $f(\alpha) := \alpha_+$ for $\alpha \in \mathbb{R}$. Then f is parabolic second-order directionally differentiable at x in the Hadamard sense and*

$$
f'(x; d) = \begin{cases} d & \text{if } x > 0, \\ d_+ & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases}
$$

and

$$
f''(x; d, w) = \begin{cases} w & \text{if } x > 0 \text{ or } x = 0, \ d > 0, \\ 0 & \text{if } x < 0 \text{ or } x = 0, \ d < 0, \\ w_+ & \text{if } x = d = 0. \end{cases}
$$

Since in the formula of the second-order directional derivative of the projection operator we will need the tangent cone and the second-order tangent set for the set $\mathcal{K}$ and its polar $\mathcal{K}^\circ$, for convenience we summarize their formulas in the following two lemmas.

LEMMA 3.3 (see [3, Lemmas 25 and 27]). *For any $x \in \mathcal{K}$, one has*

$$
T_{\mathcal{K}}(x) = \begin{cases} \mathbb{R}^m & \text{if } x \in \mathrm{int}\mathcal{K}, \\ \mathcal{K} & \text{if } x = 0, \\ \{d \in \mathbb{R}^m \mid -d_1 + \bar{x}_2^T d_2 \leq 0\} & \text{if } x \in \mathrm{bd}\mathcal{K} \backslash \{0\}. \end{cases}
$$

For any $x \in \mathcal{K}$ and $d \in T_{\mathcal{K}}(x)$,

$$
T_{\mathcal{K}}^2(x; d) = \begin{cases} \mathbb{R}^m & \text{if } d \in \mathrm{int}T_{\mathcal{K}}(x), \\ T_{\mathcal{K}}(d) & \text{if } x = 0, \\ \{w \mid w_2^T x_2 - w_1 x_1 \leq d_1^2 - \|d_2\|^2\} & \text{if } x \in \mathrm{bd}\mathcal{K} \backslash \{0\}, \ d \in \mathrm{bd}T_{\mathcal{K}}(x). \end{cases}
$$

Applying [3, Lemmas 25 and 27] to $\mathcal{K}^\circ = -\mathcal{K}$ yields the following result.

LEMMA 3.4. *For $x \in \mathcal{K}^\circ$, one has*

$$
T_{\mathcal{K}^\circ}(x) = \begin{cases} \mathbb{R}^m & \text{if } x \in \text{int}\mathcal{K}^\circ, \\ \mathcal{K}^\circ & \text{if } x = 0, \\ \{d \in \mathbb{R}^m \mid d_1 + \bar{x}_2^T d_2 \leq 0\} & \text{if } x \in \text{bd}\mathcal{K}^\circ \backslash \{0\}. \end{cases}
$$

For $x \in \mathcal{K}^\circ$ and $d \in T_{\mathcal{K}^\circ}(x)$, one has

$$
T_{\mathcal{K}^\circ}^2(x; d) = \begin{cases} \mathbb{R}^m & \text{if } d \in \text{int}T_{\mathcal{K}^\circ}(x), \\ T_{\mathcal{K}^\circ}(d) & \text{if } x = 0, \\ \{w \mid w_2^T x_2 - w_1 x_1 \leq d_1^2 - \|d_2\|^2\} & \text{if } x \in \text{bd}\mathcal{K}^\circ \backslash \{0\}, \ d \in \text{bd}T_{\mathcal{K}^\circ}(x). \end{cases}
$$

We are now ready to give the second-order directional derivative of the projection operator.

THEOREM 3.5. *The projection operator $\Pi_\mathcal{K}$ is parabolic second-order directionally differentiable in the Hadamard sense. Moreover, for any $x, d, w \in \mathbb{R}^m$, the second-order directional derivative can be calculated as in the following six cases.*
Case (i): $x \in \text{int}\mathcal{K}$. *We have $\Pi_\mathcal{K}''(x; d, w) = w$.*
Case (ii): $x \in \text{int}\mathcal{K}^\circ$. *We have $\Pi_\mathcal{K}''(x; d, w) = 0$.*
Case (iii): $x = 0$. *We have*

$$
\Pi_\mathcal{K}''(x; d, w) =
\begin{cases}
w & \text{if } d \in \text{int}\mathcal{K}, \\
0 & \text{if } d \in \text{int}\mathcal{K}^\circ, \\
\frac{1}{2}\left(\begin{array}{c} w_1 + \bar{d}_2^T w_2 \\ \left[w_1 - \frac{d_1}{\|d_2\|}\bar{d}_2^T w_2\right]\bar{d}_2 + \left[1 + \frac{d_1}{\|d_2\|}\right]w_2 \end{array} \right) & \text{if } d \in (\mathcal{K} \cup \mathcal{K}^\circ)^c, \\
w & \text{if } d \in \text{bd}\mathcal{K}\backslash\{0\}, \ w \in T_\mathcal{K}(d), \\
\frac{1}{2}\left(\begin{array}{c} w_1 + \bar{d}_2^T w_2 \\ 2w_2 + (w_1 - \bar{d}_2^T w_2)\bar{d}_2 \end{array} \right) & \text{if } d \in \text{bd}\mathcal{K}\backslash\{0\}, \ w \notin T_\mathcal{K}(d), \\
0 & \text{if } d \in \text{bd}\mathcal{K}^\circ\backslash\{0\}, \ w \in T_{\mathcal{K}^\circ}(d), \\
\frac{1}{2}(w_1 + \bar{d}_2^T w_2)\left(\begin{array}{c} 1 \\ \bar{d}_2 \end{array} \right) & \text{if } d \in \text{bd}\mathcal{K}^\circ\backslash\{0\}, \ w \notin T_{\mathcal{K}^\circ}(d), \\
\Pi_\mathcal{K}(w) & \text{if } d = 0.
\end{cases}
$$

Case (iv): $x \in \text{bd}\mathcal{K}\backslash\{0\}$. *We have*

$$
\Pi_\mathcal{K}''(x; d, w) =
\begin{cases}
w & \text{if } d \in \text{int}T_\mathcal{K}(x), \\
w & \text{if } d \in \text{bd}T_\mathcal{K}(x), \ w \in T_\mathcal{K}^2(x; d), \\
\frac{1}{2}\left(\begin{array}{c} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left[w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right]\bar{x}_2 + 2w_2 \end{array} \right) & \text{if } d \in \text{bd}T_\mathcal{K}(x), \ w \notin T_\mathcal{K}^2(x; d), \\
\frac{1}{2}\left(\begin{array}{c} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2 + 2d_1\bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 + 2w_2 + 2\frac{d_1 - \bar{x}_2^T d_2}{\|x_2\|}d_2 \end{array} \right) & \text{if } d \in T_\mathcal{K}(x)^c.
\end{cases}
$$

Case (v): $x \in \mathrm{bd}\mathcal{K}^\circ \backslash \{0\}$. *We have*

$$\Pi_{\mathcal{K}}^{''}(x; d, w) =$$

$$\begin{cases} 0 & \text{if } d \in \mathrm{int}T_{\mathcal{K}^\circ}(x), \\ 0 & \text{if } d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x), \ w \in T^2_{\mathcal{K}^\circ}(x; d), \\ \frac{1}{2}\left(w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right)\begin{pmatrix} 1 \\ \bar{x}_2 \end{pmatrix} & \text{if } d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x), w \notin T^2_{\mathcal{K}^\circ}(x; d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2 - 2d_1 \bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 + 2\frac{d_1 + \bar{x}_2^T d_2}{\|x_2\|}d_2 \end{pmatrix} & \text{if } d \in T_{\mathcal{K}^\circ}(x)^c. \end{cases}$$

Case (vi): $x \in (\mathcal{K} \cup \mathcal{K}^\circ)^c$. *We have*

$$\Pi_{\mathcal{K}}^{''}(x; d, w) =$$

$$\frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 - \frac{x_1}{\|x_2\|}\bar{x}_2^T w_2 - \frac{x_1}{\|x_2\|^2}\left(\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2\right) - 2d_1 \frac{\bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 \\ +2\frac{\|x_2\|d_1 - x_1 \bar{x}_2^T d_2}{\|x_2\|^2}d_2 + \left[1 + \frac{x_1}{\|x_2\|}\right]w_2 \end{pmatrix}.$$

Proof. By (5) and (6), the projection operator $\Pi_{\mathcal{K}}$ is the SOC function f^{soc} with $f(t) := t_+$. Applying Lemmas 3.1 and 3.2 will give the parabolic second-order directional differentiability of $\Pi_{\mathcal{K}}$ in the Hadamard sense and a formula for $\Pi_{\mathcal{K}}^{''}$. However, in some cases the formula obtained will still involve the plus operator $(\cdot)_+$. In this theorem we aim at obtaining the exact formula as proposed. For some cases, e.g., in the cases in which $x \in \mathrm{int}\mathcal{K}$; $x \in \mathrm{int}\mathcal{K}^\circ$; $x = 0$, $d \in \mathrm{int}\mathcal{K}$; $x = 0$, $d \in \mathrm{int}\mathcal{K}^\circ$; $x = 0$, $d = 0$, we can prove the results by directly using the definition of the second-order directional derivative. In some other cases, e.g., in the cases in which $x = 0$, $d \in \mathrm{bd}\mathcal{K}\backslash\{0\}$; $x = 0$, $d \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}$; $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, $d \in \mathrm{bd}T_{\mathcal{K}}(x)$; $x \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}$, $d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x)$, we can further use the representation of tangent cones in Lemmas 3.3 and 3.4 to obtain the proposed exact formula. For simplicity, we only prove some of the cases. The others can be obtained by following similar arguments.

The case in which $x \in \mathrm{int}\mathcal{K}$. In this case $\Pi_{\mathcal{K}}(x) = x$, $\Pi_{\mathcal{K}}'(x; d) = d$, and $\Pi_{\mathcal{K}}(x + td + \frac{1}{2}t^2 w) = x + td + \frac{1}{2}t^2 w$ for $t > 0$ sufficiently small. Hence

$$\Pi_{\mathcal{K}}^{''}(x; d, w) := \lim_{t \downarrow 0} \frac{\Pi_{\mathcal{K}}(x + td + \frac{1}{2}t^2 w) - \Pi_{\mathcal{K}}(x) - t\Pi_{\mathcal{K}}'(x; d)}{\frac{1}{2}t^2} = w.$$

The case in which $x = 0$ *and* $d \in \mathrm{int}\mathcal{K}$. In this case $\Pi_{\mathcal{K}}(x) = 0$ and $\Pi_{\mathcal{K}}'(x; d) = d$. Note that

$$\Pi_{\mathcal{K}}\left(x + td + \frac{1}{2}t^2 w\right) = \Pi_{\mathcal{K}}\left(td + \frac{1}{2}t^2 w\right) = td + \frac{1}{2}t^2 w$$

for $t > 0$ sufficiently small. Hence $\Pi_{\mathcal{K}}^{''}(x; d, w) = w$.

The case in which $x = 0$ *and* $d = 0$. It is obvious that $\Pi_{\mathcal{K}}(0) = 0$, $\Pi_{\mathcal{K}}'(0; 0) = 0$, and $\Pi_{\mathcal{K}}(x + td + \frac{1}{2}t^2 w) = \Pi_{\mathcal{K}}(\frac{1}{2}t^2 w) = \frac{1}{2}t^2 \Pi_{\mathcal{K}}(w)$. Hence $\Pi_{\mathcal{K}}^{''}(x; d, w) = \Pi_{\mathcal{K}}(w)$.

The case in which $x = 0$ *and* $d \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then $\Pi_{\mathcal{K}}(x) = 0$ and $d_1 = \|d_2\| \neq 0$. Directly applying Lemmas 3.1(ii) and 3.2 yield

$$(7) \quad \Pi_{\mathcal{K}}^{''}(x; d, w) = \frac{1}{2}(w_1 - \bar{d}_2^T w_2)_+ \begin{pmatrix} 1 \\ -\bar{d}_2 \end{pmatrix} + \frac{1}{2}(w_1 + \bar{d}_2^T w_2)\begin{pmatrix} 1 \\ \bar{d}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ (I - \bar{d}_2 \bar{d}_2^T)w_2 \end{pmatrix}.$$

Recall from Lemma 3.3 that $w \in T_{\mathcal{K}}(d)$ if and only if $w_1 \geq \bar{d}_2^T w_2$. It follows from (7) that

$$
\Pi_{\mathcal{K}}''(x; d, w) = \begin{cases} w & \text{if } w \in T_{\mathcal{K}}(d), \\ \dfrac{1}{2} \begin{pmatrix} w_1 + \bar{d}_2^T w_2 \\ 2w_2 + (w_1 - \bar{d}_2^T w_2)\bar{d}_2 \end{pmatrix} & \text{if } w \notin T_{\mathcal{K}}(d). \end{cases}
$$

The case in which $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $d \in \mathrm{bd}T_{\mathcal{K}}(x)$. Then $x_1 = \|x_2\| \neq 0$ and $-d_1 + \bar{x}_2^T d_2 = 0$. Directly applying Lemmas 3.1(iii) and 3.2 yield

$$
\text{(8)} \qquad \Pi_{\mathcal{K}}''(x; d, w) = \frac{1}{2}\left(w_1 - \left[\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \right] \right)_+ \begin{pmatrix} 1 \\ -\bar{x}_2 \end{pmatrix}
$$
$$
+ \frac{1}{2} \begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \right)\bar{x}_2 + 2w_2 \end{pmatrix}.
$$

Recall from Lemma 3.3 that $w \in T_{\mathcal{K}}^2(x; d)$ if and only if $w_2^T x_2 - w_1 x_1 \leq d_1^2 - \|d_2\|^2$. Hence it follows from (8) that $\Pi_{\mathcal{K}}''(x; d, w) = w$ if $w \in T_{\mathcal{K}}^2(x; d)$ and

$$
\Pi_{\mathcal{K}}''(x; d, w) = \frac{1}{2} \begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \right)\bar{x}_2 + 2w_2 \end{pmatrix}
$$

if $w \notin T_{\mathcal{K}}^2(x; d)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

4. Second-order tangent set for the SOC complementarity set. This section is devoted to deriving the exact formula for the second-order tangent set to the SOC complementarity set. To this end, we first build its connection with the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$, whose existence is guaranteed by virtue of Theorem 3.5.

PROPOSITION 4.1. *For any* $(x, y) \in \Omega$ *and* $(d, w) \in T_{\Omega}(x, y)$, *one has*

$$
T_{\Omega}^{i,2}\big((x, y); (d, w)\big) = T_{\Omega}^2\big((x, y); (d, w)\big) = \{(p, q) \mid \Pi_{\mathcal{K}}''(x - y; d - w, p - q) = p\}.
$$

Proof. Since $T_{\Omega}^{i,2}((x, y); (d, w)) \subseteq T_{\Omega}^2((x, y); (d, w))$, it suffices to show

$$
T_{\Omega}^2\big((x, y); (d, w)\big) \subseteq \Upsilon\big((x, y); (d, w)\big) \subseteq T_{\Omega}^{i,2}\big((x, y); (d, w)\big),
$$

where

$$
\Upsilon\big((x, y); (d, w)\big) := \{(p, q) \mid \Pi_{\mathcal{K}}''(x - y; d - w, p - q) = p\}.
$$

Let $(p, q) \in T_{\Omega}^2((x, y); (d, w))$. Then by definition, there exist $t_n \downarrow 0$, $(\alpha(t_n), \beta(t_n)) = o(t_n^2)$ such that $(x, y) + t_n(d, w) + \frac{1}{2}t_n^2(p, q) + (\alpha(t_n), \beta(t_n)) \in \Omega$. By the equivalence in (2), it follows that

$$
\Pi_{\mathcal{K}}\left(x - y + t_n(d - w) + \frac{1}{2}t_n^2(p - q) + \alpha(t_n) - \beta(t_n) \right)
$$
$$
= x + t_n d + \frac{1}{2}t_n^2 p + \alpha(t_n)
$$
$$
= \Pi_{\mathcal{K}}(x - y) + t_n \Pi_{\mathcal{K}}'(x - y; d - w) + \frac{1}{2}t_n^2 p + \alpha(t_n),
$$

where the last equality follows from the equivalences in (2) and (3). Therefore, $\Pi_{\mathcal{K}}''(x - y; d - w, p - q) = p$, i.e., $(p, q) \in \Upsilon\big((x, y); (d, w)\big)$.

Now, take $(p, q) \in \Upsilon((x, y); (d, w))$, i.e., $\Pi_{\mathcal{K}}''(x - y; d - w, p - q) = p$. For $t > 0$, define

$$r(t) := \Pi_{\mathcal{K}}\left(x - y + t(d - w) + \frac{1}{2}t^2(p - q)\right) - \Pi_{\mathcal{K}}(x - y) - t\Pi_{\mathcal{K}}'(x - y; d - w)$$
$$- \frac{1}{2}t^2\Pi_{\mathcal{K}}''(x - y; d - w, p - q).$$

Then $r(t) = o(t^2)$ according to the second-order directional differentiability of $\Pi_{\mathcal{K}}$ by Theorem 3.5. Note that

$$\Pi_{\mathcal{K}}\left(x - y + t(d - w) + \frac{1}{2}t^2 p + r(t) - \frac{1}{2}t^2 q - r(t)\right)$$
$$= \Pi_{\mathcal{K}}\left(x - y + t(d - w) + \frac{1}{2}t^2(p - q)\right)$$
$$= \Pi_{\mathcal{K}}(x - y) + t\Pi_{\mathcal{K}}'(x - y; d - w) + \frac{1}{2}t^2\Pi_{\mathcal{K}}''(x - y; d - w, p - q) + r(t)$$
$$= x + td + \frac{1}{2}t^2 p + r(t),$$

where the last equality follows from the equivalences in (2) and (3). This together with equivalence (2) yields that

$$\left(x + td + \frac{1}{2}t^2 p + r(t), y + tw + \frac{1}{2}t^2 q + r(t)\right) \in \Omega.$$

This means that $(p, q) \in T_{\Omega}^{i,2}((x, y); (d, w))$. The proof is complete. □

Remark 4.1. The proof of equivalence (4) in Proposition 4.1 is very similar to that of equivalence (3) in [21, Proposition 5.2]. Note that although the equivalence (3) was shown in [15, Proposition 3.1], the proof in [21, Proposition 5.2] is much more concise without going over each possible case, as in [15, Proposition 3.1]. Moreover, from the proof of [21, Proposition 5.2], one can see that the equivalence (3) can be extended to other convex cones $\mathcal{K}$ as long as the projection operator $\Pi_{\mathcal{K}}$ satisfies directional differentiability. Similarly, from the proof of Proposition 4.1, we can see that equivalence (4) in Proposition 4.1 can be extended to other convex cones $\mathcal{K}$ whenever the projection operator $\Pi_{\mathcal{K}}$ satisfies parabolic second-order directional differentiability in the Hadamard sense.

The above result tells us that for characterizing the structure of the second-order tangent set to Ω, we need to study the expression of the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$, which has been obtained in Theorem 3.5. With these preparations, the explicit expression of the second-order tangent set to Ω is given below. For convenience, we recall the formula for the tangent cone first.

LEMMA 4.2 (see [21, Theorem 5.1]).　*For any* $(x, y) \in \Omega$,

$$T_{\Omega}^i(x, y) = T_{\Omega}(x, y)$$

$$= \left\{ (d, w) \left| \begin{array}{ll} d \in \mathbb{R}^m, \ w = 0 & \text{if } x \in \text{int}\mathcal{K}, \ y = 0; \\ d = 0, \ w \in \mathbb{R}^m & \text{if } x = 0, \ y \in \text{int}\mathcal{K}; \\ x_1\hat{w} - y_1 d \in \mathbb{R}x, \ d \perp y, \ w \perp x & \text{if } x, y \in \text{bd}\mathcal{K}\backslash\{0\}; \\ d \in T_{\mathcal{K}}(x), \ w = 0 \text{ or } d \perp \hat{x}, \ w \in \mathbb{R}_+\hat{x} & \text{if } x \in \text{bd}\mathcal{K}\backslash\{0\}, \ y = 0; \\ d = 0, \ w \in T_{\mathcal{K}}(y) \text{ or } d \in \mathbb{R}_+\hat{y}, \ w \perp \hat{y} & \text{if } x = 0, \ y \in \text{bd}\mathcal{K}\backslash\{0\}; \\ d \in \mathcal{K}, \ w \in \mathcal{K}, \ d \perp w & \text{if } x = 0, \ y = 0 \end{array} \right. \right\}.$$

According to Proposition 4.1 and Lemma 4.2, we obtain the following result.

THEOREM 4.3. *The set Ω is second-order directionally differentiable at $(x, y) \in \Omega$ in direction $(d, w) \in T_\Omega(x, y)$.*

Remark 4.2. It is well known that for a convex set, the tangent cone and inner tangent cone coincide, but the inner and outer second-order tangent sets can be different; see [4, Example 3.31]. Here we show that the SOC complementarity set Ω, although it is nonconvex, is second-order directionally differentiable, i.e., the tangent cone and inner tangent cone coincide, and the inner and outer second-order tangent sets coincide as well.

The inner and outer second-order tangent sets to product sets have been studied in [4, Page 168]. In particular, for $C := C_1 \times \cdots \times C_m$ with $C_i \in \mathbb{R}^{n_i}$, at certain $x = (x_1, \ldots, x_m)$ with $x_i \in C_i$, according to [4],

$$T_C^{i,2}(x; d) = T_{C_1}^{i,2}(x_1; d_1) \times \cdots \times T_{C_m}^{i,2}(x_m; d_m)$$

and

(9) $$T_C^2(x; d) \subset T_{C_1}^2(x_1; d_1) \times \cdots \times T_{C_m}^2(x_m; d_m).$$

If all except at most one of C_i are second-order directionally differentiable, then the equality holds in (9). Noting that the second-order cone complementarity set is second-order directionally differentiable, Theorem 4.3 can then be extended to the Cartesian product of finitely many second-order cone complementarity sets.

COROLLARY 4.4. *Suppose $\Omega_1, \ldots, \Omega_l$ are all SOC complementarity sets. Then the Cartesian product $\Omega := \Omega_1 \times \Omega_2 \times \cdots \times \Omega_l$ is second-order directionally differentiable at every $(x, y) \in \Omega$ in every direction $(d, w) \in T_\Omega(x, y)$ and*

$$T_\Omega^2((x, y); (d, w)) = T_{\Omega_1}^2((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_l}^2((x_l, y_l); (d_l, w_l)).$$

Proof. Since

$$(d, w) \in T_\Omega(x, y) = T_{\Omega_1}(x_1, y_1) \times \cdots \times T_{\Omega_l}(x_l, y_l),$$

$(d_i, w_i) \in T_{\Omega_i}(x_i, y_i)$ for $i = 1, \ldots, l$. Take $(p, q) \in T_\Omega^2((x, y); (d, w))$. Hence

$$\begin{aligned} T_\Omega^2((x, y); (d, w)) &\subseteq T_{\Omega_1}^2((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_l}^2((x_l, y_l); (d_l, w_l)) \\ &= T_{\Omega_1}^{i,2}((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_l}^{i,2}((x_l, y_l); (d_l, w_l)) \\ &= T_\Omega^{i,2}((x, y); (d, w)), \end{aligned}$$

where the first inclusion and the second equation follows from [4, Page 168], and the first equation comes from Theorem 4.3. $\qquad\square$

THEOREM 4.5. *For any $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$, the formula of the second-order tangent set for the SOC complementarity set can be described as in the following six cases.*

Case (i): $x \in \mathrm{int}\mathcal{K}$ and $y = 0$. *We have $T_\Omega^2\big((x, y); (d, w)\big) = \mathbb{R}^m \times \{0\}$.*

Case (ii): $x = 0$ and $y \in \mathrm{int}\mathcal{K}$. *We have $T_\Omega^2\big((x, y); (d, w)\big) = \{0\} \times \mathbb{R}^m$.*

Case (iii): $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. *We have*

$$T_\Omega^2\big((x, y); (d, w)\big)$$
$$= \left\{ (p, q) \,\middle|\, \begin{array}{l} p \in \mathrm{bd}T_\mathcal{K}^2(x; d), \ q \in \mathrm{bd}T_\mathcal{K}^2(y; w), \\ (x_1 w_1 - y_1 d_1)\left(\frac{w_2 - w_1 \bar{y}_2}{y_1} - \frac{d_2 - d_1 \bar{x}_2}{x_1}\right) - p_1 y_2 - q_1 x_2 = x_1 q_2 + y_1 p_2 \end{array} \right\}.$$

Case (iv): $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $y = 0$. *We have*

$$T_\Omega^2\big((x,y);(d,w)\big)$$
$$= \left\{ (p,q) \left| \begin{array}{l} q = 0, \qquad\qquad\qquad\qquad\qquad\quad \text{if } d \in \mathrm{int}T_\mathcal{K}(x), \ w = 0; \\ p \in T_\mathcal{K}^2(x;d), \ q = 0, \ \text{or } p \in \mathrm{bd}T_\mathcal{K}^2(x;d), \ q \in \mathbb{R}_+\hat{x} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } d \in \mathrm{bd}T_\mathcal{K}(x), \ w = 0; \\ p \in \mathrm{bd}T_\mathcal{K}^2(x;d), \ -q_1\bar{x}_2 - 2\frac{w_1 d_2}{\|x_2\|} - 2\frac{d_1 w_2}{\|x_2\|} = q_2, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } d \perp \hat{x}, \ w \in \mathbb{R}_{++}\hat{x} \end{array} \right. \right\}.$$

Case (v): $x = 0$ *and* $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. *We have*

$$T_\Omega^2\big((x,y);(d,w)\big)$$
$$= \left\{ (p,q) \left| \begin{array}{l} p = 0, \qquad\qquad\qquad\qquad\qquad\quad \text{if } d = 0, \ w \in \mathrm{int}T_\mathcal{K}(y); \\ p = 0, \ q \in T_\mathcal{K}^2(y;w), \ \text{or } p \in \mathbb{R}_+\hat{y}, \ q \in \mathrm{bd}T_\mathcal{K}^2(y;w) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } d = 0, \ w \in \mathrm{bd}T_\mathcal{K}(y); \\ q \in \mathrm{bd}T_\mathcal{K}^2(y;w), \ -p_1\bar{y}_2 - 2\frac{w_1 d_2}{\|y_2\|} - 2\frac{d_1 w_2}{\|y_2\|} = p_2, \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } d \in \mathbb{R}_{++}\hat{y}, \ w \perp \hat{y} \end{array} \right. \right\}.$$

Case (vi): $x = y = 0$. *We have* $T_\Omega^2\big((x,y);(d,w)\big) = T_\Omega(d,w)$.

Proof. By Proposition 4.1, to describe an element $(p,q) \in T_\Omega^2((x,y);(d,w))$, it suffices to describe an element (p,q) satisfying $\Pi_\mathcal{K}''(x - y; d - w, p - q) = p$. For simplicity, we define $z := x - y$, $\xi := d - w$, and $\eta := p - q$.

Case (i): $x \in \mathrm{int}\mathcal{K}$ *and* $y = 0$. Since $z = x - y \in \mathrm{int}\mathcal{K}$, by Theorem 3.5(i), we have $\Pi_\mathcal{K}''(x - y; d - w, p - q) = p - q$. It follows that

$$\Pi_\mathcal{K}''(x - y; d - w, p - q) = p \Longleftrightarrow q = 0.$$

Hence $T_\Omega^2\big((x,y);(d,w)\big) = \mathbb{R}^m \times \{0\}$.

Case (ii): $x = 0$ *and* $y \in \mathrm{int}\mathcal{K}$. Since $z = x - y \in -\mathrm{int}\mathcal{K}$, by Theorem 3.5(ii), we know that $\Pi_\mathcal{K}''(z; d - w, p - q) = 0$. It follows that

$$\Pi_\mathcal{K}''(x - y; d - w, p - q) = p \Longleftrightarrow p = 0.$$

Hence $T_\Omega^2\big((x,y);(d,w)\big) = \{0\} \times \mathbb{R}^m$.

Case (iii): $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $x^T y = 0$. In this case $x_1 = \|x_2\| \neq 0$ and by Lemma 2.5,

$$(10) \qquad z = x - y = (x_1, x_2) - k(x_1, -x_2) = ((1-k)x_1, (1+k)x_2), \quad k = y_1/x_1.$$

This yields $z_1 + \|z_2\| = 2x_1 > 0$ and $z_1 - \|z_2\| = -2kx_1 < 0$, i.e., $z \in (\mathcal{K} \cup \mathcal{K}^\circ)^c$. Then by Theorem 3.5(vi), $\Pi_\mathcal{K}''(z; \xi, \eta) = p$, where $p = (p_1, p_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$ if and only if

$$(11) \qquad p_1 = \frac{1}{2}\left(\eta_1 + \bar{z}_2^T \eta_2 + \frac{\|\xi_2\|^2 - (\bar{z}_2^T \xi_2)^2}{\|z_2\|} \right),$$

$$p_2 = \frac{1}{2}\left(\eta_1 - \frac{z_1}{\|z_2\|}\bar{z}_2^T\eta_2 - \frac{z_1}{\|z_2\|^2}\left[\|\xi_2\|^2 - 3(\bar{z}_2^T\xi_2)^2 \right] - 2\xi_1\frac{\bar{z}_2^T\xi_2}{\|z_2\|} \right)\bar{z}_2$$

$$(12) \qquad\qquad + \frac{\|z_2\|\xi_1 - z_1\bar{z}_2^T\xi_2}{\|z_2\|^2}\xi_2 + \frac{1}{2}\left(1 + \frac{z_1}{\|z_2\|} \right)\eta_2.$$

We now try to derive equivalent expressions for (11) and (12). Since $(d, w) \in T_\Omega(x, y)$, according to Lemma 4.2, $x \perp w$, $y \perp d$, and there exists $\beta \in \mathbb{R}$ such that $x_1 \hat{w} - y_1 d = \beta x$, from which and also from $x_1 = \|x_2\| \neq 0$ we have

$$(13) \qquad\qquad w_1 = k d_1 + \beta, \quad w_2 = -k d_2 - \beta \bar{x}_2,$$

and

$$(14) \qquad\qquad \bar{x}_2^T w_2 = -w_1, \quad \bar{x}_2^T d_2 = d_1.$$

Note that $\bar{z}_2 = \bar{x}_2$ by (10). Hence it follows from (13) and (14) that

$$(15) \qquad \bar{z}_2^T \xi_2 = \bar{x}_2^T (d_2 - w_2) = d_1 + w_1 = (1 + k)d_1 + \beta,$$
$$\|\xi_2\|^2 = \|d_2 - w_2\|^2 = \|(1 + k)d_2 + \beta \bar{x}_2\|^2$$
$$= (k + 1)^2 \|d_2\|^2 + 2\beta(k + 1)d_1 + \beta^2,$$
$$(16) \qquad \xi_1 = d_1 - w_1 = (1 - k)d_1 - \beta.$$

Hence (11) can be rewritten as

$$(17) \qquad p_1 = -q_1 + \bar{x}_2^T (p_2 - q_2) + \frac{x_1 + y_1}{x_1^2}\left(\|d_2\|^2 - d_1^2\right).$$

The term in front of $\bar{z}_2$ in (12) becomes

$$\frac{1}{2}\left(\eta_1 - \frac{z_1}{\|z_2\|}\bar{z}_2^T \eta_2 - \frac{z_1}{\|z_2\|^2}\left[\|\xi_2\|^2 - 3(\bar{z}_2^T \xi_2)^2\right] - 2\xi_1 \frac{\bar{z}_2^T \xi_2}{\|z_2\|}\right)$$
$$= \frac{1}{2}\left(\eta_1 + \bar{z}_2^T \eta_2 + \frac{\|\xi_2\|^2 - (\bar{z}_2^T \xi_2)^2}{\|z_2\|}\right) - \frac{z_1 + \|z_2\|}{2\|z_2\|}\bar{z}_2^T \eta_2$$
$$+ \frac{z_1 + \|z_2\|}{2\|z_2\|^2}\left[(\bar{z}_2^T \xi_2)^2 - \|\xi_2\|^2\right] + \left[\frac{z_1}{\|z_2\|^2}(\bar{z}_2^T \xi_2)^2 - \frac{\xi_1}{\|z_2\|}(\bar{z}_2^T \xi_2)\right]$$
$$= \frac{y_1 p_1 - x_1 q_1}{x_1 + y_1} + \left[\frac{x_1 - y_1}{(x_1 + y_1)^2}(d_1 + w_1)^2 - \frac{1}{x_1 + y_1}(d_1^2 - w_1^2)\right]$$
$$= \frac{y_1 p_1 - x_1 q_1}{x_1 + y_1} + 2\frac{x_1 w_1^2 + x_1 d_1 w_1 - y_1 d_1^2 - y_1 d_1 w_1}{(x_1 + y_1)^2},$$

where the second equality uses (10), (11), and (15)–(17). It follows from (15) and (16) that the term in front of ξ_2 in (12) is

$$\frac{\xi_1}{\|z_2\|} - \frac{z_1}{\|z_2\|}\frac{\bar{z}_2^T \xi_2}{\|z_2\|} = \frac{1}{x_1 + y_1}(d_1 - w_1) - \frac{x_1 - y_1}{(x_1 + y_1)^2}(d_1 + w_1) = 2\frac{y_1 d_1 - x_1 w_1}{(x_1 + y_1)^2}.$$

The term in front of η_2 in (12) is $1/2\left(1 + (z_1/\|z_2\|)\right) = x_1/(x_1 + y_1)$. Hence (12) can be rewritten as

$$p_2 = \left(\frac{y_1 p_1 - x_1 q_1}{x_1 + y_1} + 2\frac{x_1 w_1^2 + x_1 d_1 w_1 - y_1 d_1^2 - y_1 d_1 w_1}{(x_1 + y_1)^2}\right)\bar{x}_2$$
$$(18) \qquad\qquad + 2\frac{y_1 d_1 - x_1 w_1}{(x_1 + y_1)^2}(d_2 - w_2) + \frac{x_1}{x_1 + y_1}(p_2 - q_2).$$

Further notice that $(y_1 p_1 - x_1 q_1)\bar{x}_2 = -p_1 y_2 - q_1 x_2$ and

$$
2\frac{y_1 d_1 - x_1 w_1}{(x_1 + y_1)^2}(d_2 - w_2) + 2\frac{x_1 w_1^2 + x_1 d_1 w_1 - y_1 d_1^2 - y_1 d_1 w_1}{(x_1 + y_1)^2}\bar{x}_2
$$

$$
= 2\frac{-x_1\beta}{(x_1 + y_1)^2}[(1+k)d_2 + \beta\bar{x}_2] + 2\frac{x_1\beta^2 + y_1 d_1\beta + x_1 d_1\beta}{(x_1 + y_1)^2}\bar{x}_2
$$

$$
= 2\frac{\beta}{x_1 + y_1}(-d_2 + d_1\bar{x}_2) = 2\frac{x_1 w_1 - y_1 d_1}{x_1(x_1 + y_1)}(-d_2 + d_1\bar{x}_2),
$$

where the first and third equations use (13) and the fact that $y_1 = kx_1$. Hence (18) can be rewritten as

$$
y_1 p_2 + x_1 q_2 = 2\frac{x_1 w_1 - y_1 d_1}{x_1}(-d_2 + d_1\bar{x}_2) - p_1 y_2 - q_1 x_2
$$

(19)
$$
= (x_1 w_1 - y_1 d_1)\left(\frac{w_2 - w_1\bar{y}_2}{y_1} + \frac{-d_2 + d_1\bar{x}_2}{x_1}\right) - p_1 y_2 - q_1 x_2,
$$

where the second step comes from the fact that $(-d_2 + d_1\bar{x}_2)/x_1 = (w_2 - w_1\bar{y}_2)/y_1$ due to (13). Hence (11) and (12) are equivalent to (17) and (19).

Now, multiplying (19) by $\bar{x}_2^T$ and using (14) yields

(20)
$$
x_1\bar{x}_2^T q_2 + y_1\bar{x}_2^T p_2 = y_1 p_1 - x_1 q_1.
$$

Hence it follows from (17) and (20) that

$$
p_1 = \left(1 + \frac{y_1}{x_1}\right)\bar{x}_2^T p_2 - \frac{y_1}{x_1}p_1 + \frac{x_1 + y_1}{x_1^2}(\|d_2\|^2 - d_1^2),
$$

that is,

(21)
$$
p_1 = \bar{x}_2^T p_2 + \frac{1}{x_1}(\|d_2\|^2 - d_1^2) \iff p \in \mathrm{bd}T_{\mathcal{K}}^2(x; d).
$$

Since $d_1 = (w_1 - \beta)/k$, $d_2 = -(w_2 + \beta\bar{x}_2)/k$ by (13), and $w_2^T\bar{x}_2 = -w_1$ by (14), we see that

(22)
$$
\frac{\|d_2\|^2 - d_1^2}{x_1^2} = \frac{\|w_2\|^2 - w_1^2}{y_1^2}.
$$

Similarly, it follows from (17), (20), (22), and $\bar{x}_2 = -\bar{y}_2$ that

$$
q_1 = -\frac{x_1}{y_1}q_1 + \left(1 + \frac{x_1}{y_1}\right)\bar{y}_2^T q_2 + \frac{x_1 + y_1}{y_1^2}(\|w_2\|^2 - w_1^2),
$$

that is,

(23)
$$
\frac{1}{y_1}(\|w_2\|^2 - w_1^2) = q_1 - \bar{y}_2^T q_2 \iff q \in \mathrm{bd}T_{\mathcal{K}}^2(y; w).
$$

Hence, along the line

$$
\left\{\begin{array}{l}(11),\\(12)\end{array}\right. \iff \left\{\begin{array}{l}(17),\\(19)\end{array}\right. \iff \left\{\begin{array}{l}(21),(23),\\(19),\end{array}\right.
$$

the desired result follows.

Case (iv): $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $y = 0$. In this case $z = x - y = x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

Case (iv-1): $d \in \mathrm{int}T_{\mathcal{K}}(x)$ *and* $w = 0$. Then $\xi = d - w = d \in \mathrm{int}T_{\mathcal{K}}(x)$. Hence $\Pi_{\mathcal{K}}''(z;\xi,\eta) = \eta$ by Theorem 3.5(iv). It follows that $\Pi_{\mathcal{K}}''(x - y; d - w, p - q) = p$ if and only if $q = 0$.

Case (iv-2): $d \in \mathrm{bd}T_{\mathcal{K}}(x)$ *and* $w = 0$. Then $\xi = d \in \mathrm{bd}T_{\mathcal{K}}(x)$. Hence $\Pi_{\mathcal{K}}''(z;\xi,\eta) = \Pi_{\mathcal{K}}''(x;d,\eta)$ and, by Theorem 3.5(iv),

$$\Pi_{\mathcal{K}}''(x;d,\eta)$$
$$= \begin{cases} \eta & \text{if } \eta \in T_{\mathcal{K}}^2(x;d), \\ \dfrac{1}{2}\begin{pmatrix} \eta_1 + \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(\eta_1 - \bar{x}_2^T\eta_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right)\bar{x}_2 + 2\eta_2 \end{pmatrix} & \text{if } \eta \notin T_{\mathcal{K}}^2(x;d). \end{cases}$$

Note that $\eta \in T_{\mathcal{K}}^2(x;d) \iff \eta_2^T x_2 - \eta_1 x_1 \le d_1^2 - \|d_2\|^2$ by Lemma 3.3. Hence $\Pi_{\mathcal{K}}''(x;d, p - q) = p$ if and only if either $p - q \in T_{\mathcal{K}}^2(x;d)$ and $q = 0$ or the following system holds:

$$(24) \quad \begin{cases} \eta_1 - \bar{x}_2^T\eta_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} < 0, \\ \frac{1}{2}\left(\eta_1 + \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right]\right) = p_1, \\ \frac{1}{2}\left(\eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right]\right)\bar{x}_2 + \eta_2 = p_2. \end{cases}$$

We now further simplify the system (24):

$$(24) \iff \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right] < 0, \\ \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1 + q_1, \\ \frac{1}{2}(\eta_1 - p_1 - q_1)\bar{x}_2 = q_2 \end{cases} \iff \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right] < 0, \\ \bar{x}_2^T p_2 - \bar{x}_2^T q_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1 + q_1, \\ -q_1\bar{x}_2 = q_2 \end{cases}$$

$$\iff \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right] < 0, \\ \bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1, \\ -q_1\bar{x}_2 = q_2 \end{cases} \iff \begin{cases} q_1 > 0, \\ \bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1, \\ -q_1\bar{x}_2 = q_2 \end{cases}$$

$$\iff \begin{cases} q \in \mathbb{R}_{++}\hat{x}, \\ p \in \mathrm{bd}T_{\mathcal{K}}^2(x;d). \end{cases}$$

Hence either $p \in T_{\mathcal{K}}^2(x;d)$ and $q = 0$ or $q \in \mathbb{R}_{++}\hat{x}$ and $p \in \mathrm{bd}T_{\mathcal{K}}^2(x;d)$.

Case (iv-3): $d \perp \hat{x}$ *and* $w \in \mathbb{R}_{++}\hat{x}$. Then $\bar{x}_2^T d_2 = d_1$ and $\bar{x}_2^T w_2 = -w_1$. Hence

$$(25) \qquad \xi_1 - \bar{x}_2^T\xi_2 = d_1 - w_1 - \bar{x}_2^T(d_2 - w_2) = -2w_1 < 0,$$

which implies $\xi \in T_{\mathcal{K}}(x)^c$ by Lemma 3.3. Thus by Theorem 3.5(iv), $\Pi_{\mathcal{K}}''(z;\xi,\eta) = p$ takes the form

$$(26) \qquad \begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T\eta_2 + \frac{\|\xi_2\|^2 - (\bar{x}_2^T\xi_2)^2}{\|x_2\|}\right) = p_1, \\ \frac{1}{2}\left(\eta_1 - \bar{x}_2^T\eta_2 - \frac{\|\xi_2\|^2 - 3(\bar{x}_2^T\xi_2)^2 + 2\xi_1\bar{x}_2^T\xi_2}{\|x_2\|}\right)\bar{x}_2 + \eta_2 + \frac{\xi_1 - \bar{x}_2^T\xi_2}{\|x_2\|}\xi_2 = p_2. \end{cases}$$

Note that

$$\xi_1\bar{x}_2^T\xi_2 - (\bar{x}_2^T\xi_2)^2 = (\xi_1 - \bar{x}_2^T\xi_2)\bar{x}_2^T\xi_2 = -2w_1(d_1 + w_1),$$
$$\|\xi_2\|^2 - (\bar{x}_2^T\xi_2)^2 = \|d_2 - w_2\|^2 - (d_1 + w_1)^2 = \|d_2\|^2 - d_1^2,$$

where we have used the fact that $d \perp w$ and $w_2 = -w_1 \bar{x}_2$ due to $d \perp \hat{x}$ and $w \in \mathbb{R}_{++}\hat{x}$. Therefore

$$\frac{\|\xi_2\|^2 - 3(\bar{x}_2^T \xi_2)^2 + 2\xi_1 \bar{x}_2^T \xi_2}{\|x_2\|} = \frac{\|\xi_2\|^2 - (\bar{x}_2^T \xi_2)^2}{\|x_2\|} + 2\frac{\xi_1 \bar{x}_2^T \xi_2 - (\bar{x}_2^T \xi_2)^2}{\|x_2\|}$$

$$(27) \qquad = \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} - 4\frac{w_1(d_1 + w_1)}{\|x_2\|}.$$

Putting (25) and (27) into (26) yields

$$\Pi_{\mathcal{K}}''(z; \xi, \eta) = p$$

$$\Longleftrightarrow \begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T \eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ \left(\frac{1}{2}\eta_1 - \frac{1}{2}\left[\bar{x}_2^T \eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right] + 2\frac{w_1}{\|x_2\|}(d_1 + w_1)\right)\bar{x}_2 - 2\frac{w_1}{\|x_2\|}(d_2 - w_2) = q_2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T \eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ \left(-q_1 + 2\frac{w_1}{\|x_2\|}(d_1 + w_1)\right)\bar{x}_2 - 2\frac{w_1}{\|x_2\|}(d_2 - w_2) = q_2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T \eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ -q_1 \bar{x}_2 - 2\frac{d_1 w_2}{\|x_2\|} - 2\frac{w_1 d_2}{\|x_2\|} = q_2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1, \\ -q_1 \bar{x}_2 - 2\frac{w_1 d_2}{\|x_2\|} - 2\frac{d_1 w_2}{\|x_2\|} = q_2. \end{cases}$$

Here the third equivalence uses the fact that $w_2 = -w_1 \bar{x}_2$ due to $w \in \mathbb{R}_{++}\hat{x}$ and the last step follows from substituting the expression for q_2 in the second equation into the first one to obtain

$$\bar{x}_2^T q_2 = \bar{x}_2^T \left(-q_1 \bar{x}_2 - 2\frac{w_1 d_2}{\|x_2\|} - 2\frac{d_1 w_2}{\|x_2\|}\right) = -q_1.$$

The desired result follows by noting that $p \in \text{bd}T_{\mathcal{K}}^2(x; d)$ if and only if $\bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1$ by virtue of Lemma 3.3.

Case (v): $x = 0$ and $y \in \text{bd}\mathcal{K}\backslash\{0\}$. The proof is omitted, since this case is symmetric to Case (iv).

Case (vi): $x = 0$ and $y = 0$. Since Ω is a cone, according to the definition of the second-order tangent set, we have

$$T_{\Omega}^2\big((0,0); (d, w)\big) = T_{\Omega}(d, w).$$

From all the above, the proof is complete. $\qquad\square$

5. Second-order optimality conditions for SOCMPCC. In this section, as an application of the second-order tangent set for the SOC complementarity set, we consider second-order optimality conditions for the mathematical programming with second-order cone complementarity constraints (SOCMPCC):

$$(28) \qquad\qquad \min \ f(x) \quad \text{s.t.} \ \mathcal{K} \ni G(x) \perp H(x) \in \mathcal{K},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are second-order continuously differentiable. For simplicity, we restrict our attention to the simpler case, i.e., $\mathcal{K}$ is a m-dimensional

second-order cone. All analysis can be easily carried over to more general cases where $\mathcal{K}$ is a Cartesian product of some second-order cones. SOCMPCC is an important class of optimization problems that has many applications. We refer the reader to [19, 21] and references therein for applications and the first-order necessary optimality conditions.

Define $F(x) := (G(x), H(x))$. Then SOCMPCC (28) can be rewritten as

$$(29) \qquad\qquad \min \ f(x) \quad \text{s.t.} \ \ F(x) \in \Omega.$$

For a convex set-constrained optimization problem in the form of (29), where Ω is replaced by a convex closed set K (see [4, formula (3.93)]), second-order optimality conditions that involve the second-order tangent set to K have been developed (see, e.g., [2, 4]). In particular, when the convex set K is not polyhedral, the second-order tangent set to K is needed in the second-order optimality conditions. However, if the set Ω in problem (29) is nonconvex, these optimality conditions are not applicable in general. In what follows, we will establish the second-order optimality conditions for the SOCMPCC, which is not a convex set-constrained optimization problem. We would like to emphasize that, even if the second-order cone complementarity set is nonconvex, its tangent cone and second-order tangent set have nice properties so that some of the theories in the second-order optimality conditions for a convex set-constrained optimization problem still hold. This observation relies heavily on the exact formulas for the tangent cone and second-order tangent set established in the previous section.

First we present some results needed for further analysis. Recall that the regular tangent cone is always convex. The following result shows that the regular tangent cone to the SOC complementarity set Ω is not only convex but is a subspace.

PROPOSITION 5.1. *For any* $(x, y) \in \Omega$,

$$
\begin{aligned}
\widehat{T}_\Omega(x, y) &= \mathrm{lin} T_\Omega(x, y) \\
&= \left\{ (d, w) \, \middle| \,
\begin{array}{ll}
d \in \mathbb{R}^m, \ w = 0 & \text{if } x \in \mathrm{int}\mathcal{K}, \ y = 0; \\
d = 0, \ w \in \mathbb{R}^m & \text{if } x = 0, \ y \in \mathrm{int}\mathcal{K}; \\
x_1 \hat{w} - y_1 d \in \mathbb{R}x, \ \ d \perp y, \ w \perp x & \text{if } x, y \in \mathrm{bd}\mathcal{K} \backslash \{0\}; \\
d \perp \hat{x}, \ w = 0 & \text{if } x \in \mathrm{bd}\mathcal{K} \backslash \{0\}, \ y = 0; \\
d = 0, \ w \perp \hat{y} & \text{if } x = 0, \ y \in \mathrm{bd}\mathcal{K} \backslash \{0\}; \\
d = 0, \ w = 0 & \text{if } x = 0, \ y = 0
\end{array}
\right\}.
\end{aligned}
$$

Proof. The formula of $\mathrm{lin} T_\Omega(x, y)$ is clear from that of $T_\Omega(x, y)$ in Lemma 4.2. According to the tangent-normal polarity as in Lemma 2.4, we can obtain the formula of $\widehat{T}_\Omega(x, y)$ by taking the polar of the limiting normal cone to Ω given in [20, Theorem 5.1]. The obtained formula of $\mathrm{lin} T_\Omega(x, y)$ and $\widehat{T}_\Omega(x, y)$ shows that they have the same expression. $\square$

The exact formulas established in Theorem 4.5 and Proposition 5.1 immediately imply the following results.

COROLLARY 5.2. *For* $(x, y) \in \Omega$, $T_\Omega(x, y) + \widehat{T}_\Omega(x, y) = T_\Omega(x, y)$.

PROPOSITION 5.3. *For* $(x, y) \in \Omega$ *and* $(d, w) \in T_\Omega(x, y)$,

$$T_\Omega^2\big((x, y); (d, w)\big) + \widehat{T}_\Omega(x, y) = T_\Omega^2\big((x, y); (d, w)\big).$$

Proof. The inclusion "$\supseteq$" is clear, since $(0, 0) \in \widehat{T}_\Omega(x, y)$. For all cases except where $x, y \in \mathrm{bd}\mathcal{K} \backslash \{0\}$, it is easy to see that "$\subseteq$" can be achieved by using the formula

of $T_\Omega^2((x,y);(d,w))$ given in Theorem 4.5 and the formula of $\widehat{T}_\Omega(x,y)$ given in Proposition 5.1. Now consider the case where $x,y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Let $(p,q) \in T_\Omega^2((x,y);(d,w))$ and $(u,v) \in \widehat{T}_\Omega(x,y)$. Since $p \in \mathrm{bd}T_\mathcal{K}^2(x;d)$, we have $\hat{x}^T p = \|d_2\|^2 - d_1^2$ by Lemma 3.3. Hence $\hat{x}^T(p+u) = \hat{x}^T p + \hat{x}^T u = \|d_2\|^2 - d_1^2$ due to the fact that $u \perp \hat{x}$ (since $u \perp y$ by Proposition 5.1 and $y \in \mathbb{R}_{++}\hat{x}$). This means $p+u \in \mathrm{bd}T_\mathcal{K}^2(x;d)$. Similarly, we can obtain $q+v \in \mathrm{bd}T_\mathcal{K}^2(y;w)$. Since $(u,v) \in \widehat{T}_\Omega(x,y)$, it follows from Proposition 5.1 that there exists $\tau \in \mathbb{R}$ such that $x_1\hat{v} - y_1 u = \tau x$. Thus

$$(30) \qquad x_1 v_2 + y_1 u_2 = -\tau x_2 = -\frac{x_1 v_1 - y_1 u_1}{x_1} x_2 = -v_1 x_2 - u_1 y_2,$$

where the last step comes from Lemma 2.5. Since we have $x,y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $(p,q) \in T_\Omega^2((x,y);(d,w))$, it follows from Theorem 4.5 that

$$(31) \qquad (p,q) \in T_\Omega^2\big((x,y);(d,w)\big) \Longleftrightarrow \left\{ \begin{array}{l} p \in \mathrm{bd}T_\mathcal{K}^2(x;d), \ q \in \mathrm{bd}T_\mathcal{K}^2(y;w), \\ \xi - p_1 y_2 - q_1 x_2 = x_1 q_2 + y_1 p_2, \end{array} \right.$$

where

$$\xi := (x_1 w_1 - y_1 d_1)\left(\frac{w_2 - w_1 \bar{y}_2}{y_1} - \frac{d_2 - d_1 \bar{x}_2}{x_1}\right).$$

This, together with (30), implies

$$\begin{aligned} x_1(q_2 + v_2) + y_1(p_2 + u_2) &= x_1 q_2 + y_1 p_2 - v_1 x_2 - u_1 y_2 \\ &= \xi - p_1 y_2 - q_1 x_2 - v_1 x_2 - u_1 y_2 \\ &= \xi - (p_1 + u_1)y_2 - (q_1 + v_1)x_2. \end{aligned}$$

Hence together with $p + u \in \mathrm{bd}T_\mathcal{K}^2(x;d)$ and $q + v \in \mathrm{bd}T_\mathcal{K}^2(y;w)$, we have that $(p+u,q+v) \in T_\Omega^2((x,y);(d,w))$ by virtue of (31). $\square$

With these preparations, we are now ready to develop a second-order necessary optimality condition for SOCMPCC. Define the Lagrange function as $L(x,\lambda) := f(x) + \langle F(x), \lambda \rangle$ and define the following three multiplier sets:

$$\begin{aligned} \Lambda^c(x^*) &:= \{\lambda \mid \nabla_x L(x^*,\lambda) = 0, \ \lambda \in N_\Omega^c(F(x^*))\}, \\ \Lambda(x^*) &:= \{\lambda \mid \nabla_x L(x^*,\lambda) = 0, \ \lambda \in N_\Omega(F(x^*))\}, \\ \Lambda^F(x^*) &:= \{\lambda \mid \nabla_x L(x^*,\lambda) = 0, \ \lambda \in \widehat{N}_\Omega(F(x^*))\}. \end{aligned}$$

Denote by $C(x^*) := \big\{d \mid \nabla f(x^*)d \le 0, \ \nabla F(x^*)d \in T_\Omega(F(x^*))\big\}$ the critical cone. Note that if there exists $\lambda \in \Lambda^F(x^*)$, then

$$C(x^*) = \big\{d \mid \nabla f(x^*)d = 0, \ \nabla F(x^*)d \in T_\Omega(F(x^*))\big\}.$$

THEOREM 5.4. *Let x^* be a locally optimal solution of SOCMPCC. Suppose that the nondegeneracy condition*

$$(32) \qquad\qquad \nabla F(x^*)\mathbb{R}^n + \mathrm{lin}T_\Omega(F(x^*)) = \mathbb{R}^{2m}$$

holds. Then $\Lambda^c(x^*) = \Lambda(x^*) = \Lambda^F(x^*) = \{\lambda_0\}$ and

$$\nabla_{xx}^2 L(x^*,\lambda_0)(d,d) - \sigma\big(\lambda_0|T_\Omega^2(F(x^*);\nabla F(x^*)d)\big) \ge 0 \quad \forall d \in C(x^*).$$

Proof. Step 1. We prove $\Lambda^c(x^*) = \Lambda(x^*) = \Lambda^F(x^*) = \{\lambda_0\}$. Since $\nabla F(x^*)\mathbb{R}^n +$ $\lim T_\Omega(F(x^*)) = \nabla F(x^*)\mathbb{R}^n + \widehat{T}_\Omega(F(x^*)) = \mathbb{R}^{2m}$, taking the polar on both sides of the above equation, by the rule for polar cones [18, Corollary 11.25] and the fact that $(\widehat{T}_\Omega)^\circ = N^c_\Omega$, we have

$$(33) \qquad \nabla F(x^*)^T \lambda = 0, \ \lambda \in N^c_\Omega(F(x^*)) \quad \Longrightarrow \quad \lambda = 0.$$

Suppose that $\lambda^1, \lambda^2 \in \Lambda^c(x^*)$. Then $\lambda^1 - \lambda^2$ satisfies $\nabla F(x^*)^T(\lambda^1 - \lambda^2) = 0$ and $\lambda^1 - \lambda^2 \in N^c_\Omega(F(x^*))$ since $N^c_\Omega(F(x))$ is a subspace (because $\widehat{T}_\Omega(x, y)$ is a subspace and $N^c_\Omega = (\widehat{T}_\Omega)^\circ$). Thus $\lambda^1 = \lambda^2$ by (33). This means that $\Lambda^c(x^*)$ is a singleton. Since $\Lambda^F(x^*) \subseteq \Lambda(x^*) \subseteq \Lambda^c(x^*)$, it remains to show that $\Lambda^F(x^*)$ is nonempty. Since $N_\Omega \subseteq N^c_\Omega$, condition (33) ensures that

$$(34) \qquad \nabla F(x^*)^T \lambda = 0, \ \lambda \in N_\Omega(F(x^*)) \quad \Longrightarrow \quad \lambda = 0,$$

which in turn implies that the system $F(x) - \Omega$ is metrically regular at $(x^*, 0)$. Thus, according to [12, Theorem 4], Proposition 5.1, and Corollary 5.2, we have $\widehat{N}_S(x) = \nabla F(x)^T \widehat{N}_\Omega(F(x))$, where $S := \{x \mid F(x) \in \Omega\}$. As x^* is a local optimal solution of problem (29), we have $0 \in \nabla f(x^*)^T + \widehat{N}_S(x^*) = \nabla f(x^*)^T + \nabla F(x^*)^T \widehat{N}_\Omega(F(x^*))$, which indicates that $\Lambda^F(x^*)$ is nonempty. Hence $\Lambda^c(x^*), \Lambda(x^*), \Lambda^F(x^*)$ are all singletons and coincide with each other. Let us denote the unique element by λ_0.

Step 2. We show that for all $d \in C(x^*)$ and for any convex subset $\mathcal{T}(d)$ in $T^2_\Omega(F(x^*); \nabla F(x^*)d)$, $\nabla^2_{xx} L(x^*, \lambda_0)(d, d) - \sigma(\lambda_0 | \mathcal{T}(d)) \geq 0$. The idea of the proof is inspired by the arguments in [2, Theorem 3.1] and using the properties of the tangent cone and second-order tangent set discussed above. For the sake of completeness, we give the detailed proof here. Consider the set $\Gamma(d) := \text{cl}\{\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\}$. Since the regular tangent cone is convex, the set $\Gamma(d)$ is closed and convex. Moreover, it follows from Proposition 5.3 and the fact the second-order tangent set is closed that $\Gamma(d) \subseteq T^2_\Omega(F(x^*); \nabla F(x^*)d)$. Because x^* is a locally optimal solution of problem (29), by definition of the second-order tangent cone, we can show that

$$\nabla f(x^*)w + \nabla^2 f(x^*)(d, d) \geq 0 \quad \forall d \in C(x^*), \ w \in T^2_S(x^*; d).$$

Since (34) holds, by [18, Proposition 13.13], the chain rule for tangent sets (1) holds with Θ taken as Ω. It follows that for all $d \in C(x^*)$, the optimization problem

$$\begin{cases} \min\limits_w & \nabla f(x^*)w + \nabla^2 f(x^*)(d, d) \\ \text{s.t.} & \nabla F(x^*)w + \nabla^2 F(x^*)(d, d) \in T^2_\Omega\big(F(x^*); \nabla F(x^*)d\big) \end{cases}$$

has nonnegative optimal value. Since $\Gamma(d) \subseteq T^2_\Omega(F(x^*); \nabla F(x^*)d)$, it is clear that the convex set constrained problem

$$(35) \qquad \begin{cases} \min\limits_w & \nabla f(x^*)w + \nabla^2 f(x^*)(d, d) \\ \text{s.t.} & \nabla F(x^*)w + \nabla^2 F(x^*)(d, d) \in \Gamma(d) \end{cases}$$

has nonnegative optimal value as well. Since the optimization problem (35) can be put into the form of problem [4, formula (2.291)] involving an indicator function of set $\Gamma(d)$ and the dual problem of [4, formula (2.291)] is in the form of [4, formula (2.298)], and the conjugate function of an indicator function is the support function, the dual problem of (35) is

$$\max\limits_\lambda \left\{ \inf\limits_w \mathcal{L}(w, \lambda) - \sigma\left(\lambda | \Gamma(d)\right) \right\},$$

where $\mathcal{L}(w, \lambda) := \nabla_x L(x^*, \lambda)w + \nabla_{xx}^2 L(x^*, \lambda)(d, d)$ is the Lagrange function of (35). Note that

$$\sigma(\lambda|\Gamma(d)) = \sigma\big(\lambda|\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\big) = \sigma(\lambda|\mathcal{T}(d)) + \sigma\big(\lambda|\widehat{T}_\Omega(F(x^*))\big) = +\infty$$

whenever $\lambda \notin [\widehat{T}_\Omega(F(x))]^\circ = N_\Omega^c(F(x))$. Therefore, the dual problem of (35) is

$$(36) \quad \max_{\lambda \in \Lambda^c(x^*)} \big\{\nabla_{xx}^2 L(x^*, \lambda)(d, d) - \sigma\left(\lambda|\Gamma(d)\right)\big\} = \nabla_{xx}^2 L(x^*, \lambda_0)(d, d) - \sigma\left(\lambda_0|\Gamma(d)\right),$$

where the equality holds since $\Lambda^c(x^*) = \{\lambda_0\}$ by Step 1.

Since $\lim T_\Omega(F(x^*)) = \widehat{T}_\Omega(F(x^*))$ by Proposition 5.1 and $\lim T_\Omega(F(x^*))$ is a subspace, we have $\lim T_\Omega(F(x^*)) = -\widehat{T}_\Omega(F(x^*))$. Hence condition (32) is $\nabla F(x^*)\mathbb{R}^n - \widehat{T}_\Omega(F(x^*)) = \mathbb{R}^{2m}$, which in turn implies $\nabla F(x^*)\mathbb{R}^n - \big(\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\big) = \mathbb{R}^{2m}$. Hence $\nabla F(\bar{x})\mathbb{R}^n - \Gamma(d) = \mathbb{R}^{2m}$. So the Robinson's constraint qualification (see [4, formula (2.313)]) for problem (35) holds. It ensures that the zero dual gap property holds (see [4, Theorem 2.165]). Hence the optimal value of the dual problem (36) is equal to the optimal value of problem (35) and hence nonnegative. In addition, noting that $\mathcal{T}(d) \subseteq \Gamma(d)$, $\sigma(\lambda_0|\mathcal{T}(d)) \leq \sigma(\lambda_0|\Gamma(d))$, which further implies that

$$(37) \qquad\qquad \nabla_{xx}^2 L(x^*, \lambda_0)(d, d) - \sigma\left(\lambda_0|\mathcal{T}(d)\right) \geq 0.$$

Step 3. Note that $T_\Omega^2(F(x^*); \nabla F(x^*)d) = \bigcup_{a \in T_\Omega^2(F(x^*); \nabla F(x^*)d)}\{a\}$ is the union of convex sets. For each $a \in T_\Omega^2(F(x^*); \nabla F(x^*)d)$, by (37) we have

$$\nabla_{xx}^2 L(x^*, \lambda_0)(d, d) - \langle \lambda_0, a \rangle \geq 0.$$

This then yields the desired result

$$\nabla_{xx}^2 L(x^*, \lambda_0)(d, d) - \sigma\left(\lambda_0|T_\Omega^2(F(x^*); \nabla F(x^*)d)\right) \geq 0. \qquad\qquad \square$$

Remark 5.1. The nondegeneracy condition (32), together with the special geometric structure of the second-order cone complementarity set, can ensure not only the uniqueness of Lagrangian multiplier in Step 1, but also the zero-dual gap property between (35) and (36) in Step 2. The nondegeneracy condition, stronger than the Robinson's constraint qualification, is a generalization of the linear independence constraint qualification in the conic case. We refer the reader to [4, Proposition 4.75] for a detailed discussion on the relationship between the nondegeneracy condition and the uniqueness of multiplier in the convex case.

We next derive the exact formula for the support function of the second-order tangent set to the SOC complementarity set needed in applying Theorem 5.4. Under the assumption of Theorem 5.4 we have

$$C(x^*) = \big\{d \mid \nabla f(x^*)d = 0, \ \nabla F(x^*)d \in T_\Omega(F(x^*))\big\}.$$

Thus $d \in C(x^*)$ if and only if $\nabla F(x^*)d \in T_\Omega(F(x^*))$ and $\langle \lambda_0, \nabla F(x^*)d \rangle = 0$. Therefore the following results will be useful.

PROPOSITION 5.5. *For* $(x, y) \in \Omega$ *and* $(d, w) \in T_\Omega(x, y)$, *take* $(u, v) \in \widehat{N}_\Omega(x, y)$ *such that* $\langle (u, v), (d, w) \rangle = 0$. *Then*

$$\sigma\big((u, v)|T_\Omega^2((x, y); (d, w))\big) = \left\{ \begin{array}{ll} 0 & \text{if } x \in \text{int}\mathcal{K}, \ y = 0; \\ 0 & \text{if } x = 0, \ y \in \text{int}\mathcal{K}; \\ 0 & \text{if } x = 0, \ y = 0 \end{array} \right\}.$$

If $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$, then

$$\sigma\big((u,v)|T_\Omega^2((x,y);(d,w))\big)$$

$$= \left\{ \begin{array}{ll} 0 & \text{if } d \in \mathrm{int}T_\mathcal{K}(x), \ w = 0; \\ -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) - 2\frac{w_1 d_2^T v_2}{\|x_2\|} - 2\frac{d_1 w_2^T v_2}{\|x_2\|} & \text{if } d \in \mathrm{bd}T_\mathcal{K}(x), \ w \in \mathbb{R}_+\hat{x} \end{array} \right\}.$$

If $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, then

$$\sigma\big((u,v)|T_\Omega^2((x,y);(d,w))\big)$$

$$= \left\{ \begin{array}{ll} 0 & \text{if } d = 0, \ w \in \mathrm{int}T_\mathcal{K}(y); \\ -\frac{v_1}{y_1}(w_1^2 - \|w_2\|^2) - 2\frac{d_1 w_2^T u_2}{\|y_2\|} - 2\frac{w_1 d_2^T u_2}{\|y_2\|} & \text{if } d \in \mathbb{R}_+\hat{y}, \ w \in \mathrm{bd}T_\mathcal{K}(y) \end{array} \right\}.$$

If $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, then

$$\sigma\big((u,v)|T_\Omega^2((x,y);(d,w))\big) = \frac{x_1 u_1 + y_1 v_1}{x_1^2}\left(\|d_2\|^2 - d_1^2\right) + \frac{x_1 w_1 - y_1 d_1}{x_1 y_1}\left(w^T v - d^T u\right).$$

Proof. For $(x,y) \in \Omega$, take $(d,w) \in T_\Omega(x,y)$, $(p,q) \in T_\Omega^2((x,y);(d,w))$ with the exact formula given in Theorem 4.5 and $(u,v) \in \widehat{N}_\Omega(x,y)$, whose exact formula can be found in [20, Theorem 3.1].

Case (i): $x \in \mathrm{int}\mathcal{K}$ *and* $y = 0$. In this case $u = 0$ and $q = 0$. Hence

$$\sigma\big((u,v)|T_\Omega^2((x,y);(d,w))\big) = \max\{\langle(u,v),(p,q)\rangle \mid (p,q) \in T_\Omega^2((x,y);(d,w))\} = 0.$$

The proof for the case in which $x = 0$ and $y \in \mathrm{int}\mathcal{K}$ is similar and hence we omit it.

Case (ii): $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $y = 0$. Then $u \in \mathbb{R}_-\hat{x}$ and $v \in \hat{x}^\circ$ by the formula of $\widehat{N}_\Omega(x,y)$.

Case (ii-1): $d \in \mathrm{int}T_\mathcal{K}(x)$ *and* $w = 0$. Then by Theorem 4.5, $q = 0$. Since $0 = \langle u,d\rangle + \langle v,w\rangle = \langle u,d\rangle$ and $d \in \mathrm{int}T_\mathcal{K}(x)$ (i.e., $d^T\hat{x} > 0$), we have $u = 0$. Hence

$$\sigma\big((u,v)|T_\Omega^2((x,y);(d,w))\big) = \max\{\langle(u,v),(p,q)\rangle|(p,q) \in T_\Omega^2((x,y);(d,w))\} = 0.$$

Case (ii-2): $d \in \mathrm{bd}T_\mathcal{K}(x)$ *and* $w = 0$. Then $q = 0$ or $q \in \mathbb{R}_+\hat{x}$ by Theorem 4.5. Hence

$$\sigma\big((u,v)|T_\Omega^2\big((x,y);(d,w)\big)\big) = \max\left\{\sigma(u|T_\mathcal{K}^2(x;d)), \sigma(u|\mathrm{bd}T_\mathcal{K}^2(x;d)) + \sigma(v|\mathbb{R}_+\hat{x})\right\}$$

$$= \sigma(u|T_\mathcal{K}^2(x;d)) = -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2),$$

where the second equality holds because $\sigma(v|\mathbb{R}_+\hat{x}) = 0$ since $v \in \hat{x}^\circ$, and the last step comes from the fact that since $u \in \mathbb{R}_-\hat{x}$, $\langle u,p\rangle = \frac{u_1}{x_1}\langle\hat{x},p\rangle \le \frac{u_1}{x_1}(\|d_2\|^2 - d_1^2)$ for all $p \in T_\mathcal{K}^2(x;d)$ by Lemma 3.3, and the maximum can be attained by letting $p = \frac{\|d_2\|^2 - d_1^2}{2x_1^2}\hat{x}$.

Case (ii-3): $d \in \mathrm{bd}T_\mathcal{K}(x)$ *and* $w \in \mathbb{R}_{++}\hat{x}$. From the formula for $\mathrm{bd}T_\mathcal{K}(x)$ in this case, we get $d \perp \hat{x}$. Hence $\langle v,w\rangle = \langle(u,v),(d,w)\rangle = 0$, taking into account that $u \in \mathbb{R}_-\hat{x}$. This further implies that $v \perp \hat{x}$ (i.e., $v_1 = \bar{x}_2^T v_2$), because $w \in \mathbb{R}_{++}\hat{x}$.

Hence

$$
\begin{aligned}
\sigma\big((u,v)&|T_\Omega^2\big((x,y);(d,w)\big)\big)\\
&= \sigma(u|\mathrm{bd}T_\mathcal{K}^2(x;d)) + \langle v,q\rangle\\
&= -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) + v_1 q_1 - q_1 v_2^T \bar{x}_2 - 2\frac{w_1 d_2^T v_2}{\|x_2\|} - 2\frac{d_1 w_2^T v_2}{\|x_2\|}\\
&= -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) - 2\frac{w_1 d_2^T v_2}{\|x_2\|} - 2\frac{d_1 w_2^T v_2}{\|x_2\|}.
\end{aligned}
$$

Case (iii): $x = 0$ *and* $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. The argument is similar to the above case.

Case (iv): $x,y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Note that in this case, since $x^T y = 0$, we have $y = k\hat{x}$ with $k := y_1/x_1$. Since $(u,v) \in \widehat{N}_\Omega(x,y)$ and $(d,w) \in T_\Omega(x,y)$, by the formulas of $\widehat{N}_\Omega(x,y)$ and $T_\Omega(x,y)$ we have $v \perp y$, $d \perp y$, and there exist $\beta, \gamma \in \mathbb{R}$ such that $\hat{u} + kv = \beta x$ and $\hat{w} - kd = rx$. To simplify the notation, let

$$
\xi := (x_1 w_1 - y_1 d_1)\left(\frac{w_2 + w_1 \bar{x}_2}{y_1} - \frac{d_2 - d_1 \bar{x}_2}{x_1}\right).
$$

Since $v \perp y$, $y \in \mathbb{R}\hat{x}$, and $x_1 = \|x_2\| \neq 0$, we have $v_1 - \bar{x}_2^T v_2 = 0$. It follows that
(38)
$$
v_2^T \xi = (x_1 w_1 - y_1 d_1)\left(\frac{w_2^T v_2 + w_1 v_1}{y_1} - \frac{d_2^T v_2 - d_1 v_1}{x_1}\right) = \frac{x_1 w_1 - y_1 d_1}{y_1}(w^T v - d^T u),
$$

where in the last step we used the fact that $d^T \hat{v} = (1/k)d^T(\beta\hat{x} - u) = -(1/k)d^T u$ since $d \perp \hat{x}$. By the formula of $T_\Omega^2\big((x,y);(d,w)\big)$ in Theorem 4.5, for this case we have

(39) $\qquad p \in \mathrm{bd}T_\mathcal{K}^2(x;d), \quad q \in \mathrm{bd}T_\mathcal{K}^2(y;w), \quad \xi - p_1 y_2 - q_1 x_2 = x_1 q_2 + y_1 p_2.$

Therefore,

$$
\begin{aligned}
\langle u,p\rangle &+ \langle v,q\rangle\\
&= \langle \hat{u},\hat{p}\rangle + \langle v,q\rangle = \langle \beta x - kv,\hat{p}\rangle + \langle v,q\rangle = \beta\langle\hat{x},p\rangle + \langle v,q - k\hat{p}\rangle\\
&= \beta\langle\hat{x},p\rangle + v_1(q_1 - kp_1) + v_2^T\left(\frac{\xi}{x_1} + (kp_1 - q_1)\bar{x}_2\right)\\
&= \beta\langle\hat{x},p\rangle + \frac{1}{x_1}v_2^T\xi = \frac{x_1 u_1 + y_1 v_1}{x_1^2}\big(\|d_2\|^2 - d_1^2\big) + \frac{x_1 w_1 - y_1 d_1}{x_1 y_1}(w^T v - d^T u),
\end{aligned}
$$

where the fourth equality holds by virtue of (39), the fifth equality holds because $v_1 = v_2^T \bar{x}_2$, and the sixth equality holds due to (38) and (39). The desired formula follows.

Case (v): $x = 0$ *and* $y = 0$. In this case $d, w \in \mathcal{K}$, $d \perp w$, $(u,v) \in \widehat{N}_\Omega(x,y) = (-\mathcal{K}, -\mathcal{K})$, and $(p,q) \in T_\Omega^2\big((x,y);(d,w)\big) = T_\Omega(d,w)$.

Case (v-1): $d = 0$ *and* $w \in \mathrm{int}\mathcal{K}$. Since $\langle v,w\rangle = \langle(u,v),(d,w)\rangle = 0$ and $v \in -\mathcal{K}$, we have $v = 0$. Since $d = 0$ and $w \in \mathrm{int}\mathcal{K}$, $(p,q) \in T_\Omega^2\big((x,y);(d,w)\big) = T_\Omega(d,w)$ implies that $p = 0$. Hence $\langle(u,v),(p,q)\rangle = 0$. It follows that $\sigma\big((u,v)|T_\Omega^2\big((x,y);(d,w)\big)\big) = 0$.

Case (v-2): $d \in \mathrm{int}\mathcal{K}$ *and* $w = 0$. This is similar to the above case.

Case (v-3): $d, w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then since $(u,v) \in (-\mathcal{K}, -\mathcal{K})$ and $\langle(u,v),(d,w)\rangle = 0$, we have $u \in \mathbb{R}_-\hat{d} = \mathbb{R}_- w$ and $v \in \mathbb{R}_-\hat{w} = \mathbb{R}_- d$. Since $(p,q) \in T_\Omega(d,w)$ and $d, w \in$

$\mathrm{bd}\mathcal{K}\backslash\{0\}$, we have $p \perp w$ and $q \perp d$. Hence $p \perp u$ and $q \perp v$. So $\langle(u,v),(p,q)\rangle = 0$. It follows that $\sigma((u,v)|T_\Omega^2((x,y);(d,w))) = 0$.

Case (v-4): $d = 0$ *and* $w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Since $\langle v, w\rangle = \langle(u,v),(d,w)\rangle = 0$ and $v \in -\mathcal{K}$, we have $v \in \mathbb{R}_-\hat{w}$. In this case, since $(p,q) \in T_\Omega(d,w)$ with $d = 0$ and $w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, we have either $p = 0$ and $q \in T_\mathcal{K}(w)$ or $p \in \mathbb{R}_+\hat{w}$ and $q \perp \hat{w}$. If $p = 0$ and $q \in T_\mathcal{K}(w)$ (i.e., $\hat{w}^T q \geq 0$), then $\langle(u,v),(p,q)\rangle = \langle v, q\rangle \leq 0$ and the maximum is 0, which can be attained by letting $q = 0$. If $p \in \mathbb{R}_+\hat{w}$ and $q \perp \hat{w}$, then $\langle(u,v),(p,q)\rangle = \langle u, p\rangle \leq 0$, where the last step is due to $u \in -\mathcal{K}$ and $p \in \mathbb{R}_+\hat{w} \in \mathcal{K}$, and the maximum is 0, which can be attained by letting $p = 0$. It follows that $\sigma((u,v)|T_\Omega^2((x,y);(d,w))) = 0$.

Case (v-5): $d \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $w = 0$. This is similar to the above case by an analogous argument.

Case (v-6): $d = 0$ *and* $w = 0$. In this case $(p,q) \in T_\Omega(d,w) = \Omega$. Since $(u,v) \in (-\mathcal{K}, -\mathcal{K})$, we have $\langle(u,v),(p,q)\rangle \leq 0$ and the maximum is 0, which can be attained by letting $(p,q) = (0,0)$. It follows that $\sigma((u,v)|T_\Omega^2((x,y);(d,w))) = 0$. □

Example 5.1. Consider the following SOCMPCC:

$$\min \quad f(x) := -x_2^2 + x_1 - x_4$$
$$\text{s.t.} \quad \mathcal{K} \ni G(x) := (x_1, x_3 - x_1, x_1 - x_2) \perp (-x_2 + 1, x_1, x_1 - x_4) =: H(x) \in \mathcal{K}.$$

Since $x_1 \geq x_1 - x_2$ and $-x_2 + 1 \geq 0$, we have $x_2 \in [0,1]$. Hence $x_2^2 \leq x_2$. Since $-x_2 + 1 \geq -x_1 + x_4$, $-x_2 + x_1 - x_4 \geq -1$. Thus $-x_2^2 + x_1 - x_4 \geq -x_2 + x_1 - x_4 \geq -1$. Hence $x^* = (0,0,0,1)$ is an optimal solution, and $G(x^*) = (0,0,0)$ and $H(x^*) = (1,0,-1) \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

Note that

$$\nabla G(x^*) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad \nabla H(x^*) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix},$$

and by the formula of the tangent cone in Lemma 4.2, we have
(40)
$$T_\Omega(G(x^*), H(x^*)) = \left\{(d,w) \left| \begin{array}{l} \text{either } d = 0, \ -w_1 - w_3 \leq 0 \\ \text{or } d = t(1,0,1) \text{ for some } t \geq 0, \ w_1 + w_3 = 0 \end{array} \right. \right\}.$$

Hence
$$\mathrm{lin}T_\Omega(G(x^*), H(x^*)) = \{((0,0,0),(\tau_1, \tau_2, -\tau_1)) \mid \tau_1, \tau_2 \in \mathbb{R}\}.$$

For any $v \in \mathbb{R}^6$, take

$$\xi = (v_1, v_1 - v_3, v_1 + v_2, v_3 - v_4 - v_6) \in \mathbb{R}^4$$

and

$$\tau = (v_1 - v_3 + v_4, v_5 - v_1, -(v_1 - v_3 + v_4)) \in \mathbb{R}^3.$$

Then
$$v = \begin{pmatrix} \nabla G(x^*) \\ \nabla H(x^*) \end{pmatrix} \xi + \begin{pmatrix} 0 \\ \tau \end{pmatrix} \in \nabla F(x^*)\mathbb{R}^4 + \mathrm{lin}T_\Omega(F(x^*)).$$

Since v is arbitrarily taken from $\mathbb{R}^6$, condition (32) holds.

The Lagrangian multiplier system is

$$
\begin{cases}
\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + \lambda_1^G \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^G \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3^G \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \\[2em]
\quad + \lambda_1^H \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2^H \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_3^H \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\[2em]
\quad (\lambda^G, \lambda^H) \in N_\Omega\big(G(x^*), H(x^*)\big).
\end{cases}
$$

Since $G(x^*) = (0,0,0)$ and $H(x^*) = (1,0,-1) \in \mathrm{bd}\mathcal{K} \setminus \{0\}$, we obtain the following expression of the limiting normal cone from [20, Theorem 5.1]:

$$
N_\Omega\big(G(x^*), H(x^*)\big) = \left\{ (u,v) \;\middle|\; \begin{array}{l} v = 0 \text{ or } u_1 + u_3 = 0, \ v = t(1,0,1), \ t \in \mathbb{R}, \\ \text{or } u_1 + u_3 \le 0, \ v = t(1,0,1), \ t \le 0 \end{array} \right\}.
$$

Hence the only multipliers (λ^G, λ^H) satisfying the Lagrangian multiplier system are $\lambda^G = (-1,0,1)$ and $\lambda^H = (-1,0,-1)$. Note that

$$
\begin{aligned}
C(x^*) &= \{ d \in \mathbb{R}^4 \mid (d_1, d_3 - d_1, d_1 - d_2, -d_2, d_1, d_1 - d_4) \in T_\Omega(G(x^*), H(x^*)), \ d_1 \le d_4 \} \\
&= \{ d = (t,0,t,t) \mid t \ge 0 \},
\end{aligned}
$$

where the second equality follows from (40).

Since $\nabla G(x^*)d = (t,0,t)$, $\nabla H(x^*)d = (0,t,0)$ for any $d = (t,0,t,t)$ with $t \ge 0$ in $C(x^*)$, by Proposition 5.5 we obtain

$$
\sigma\big((\lambda^G, \lambda^H) | T_\Omega^2(G(x^*), H(x^*); \nabla G(x^*)d, \nabla H(x^*)d)\big) = -t^2 = -d_1^2.
$$

Since

$$
\nabla_{xx}^2 L(x, \lambda) = \nabla^2 f(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

we have

$$
(41) \qquad \nabla_{xx}^2 L(x^*, \lambda)(d,d) = 0 \quad \forall d \in C(x^*),
$$

and by Theorem 5.4,

$$
\begin{aligned}
\Upsilon(x^*, \lambda)(d) \\
:= \nabla_{xx}^2 L(x^*, \lambda)(d,d) - \sigma\big((\lambda^G, \lambda^H) | T_\Omega^2(G(x^*), H(x^*); \nabla G(x^*)d, \nabla H(x^*)d)\big) \\
(42) \qquad = d_1^2 \ge 0 \quad \forall d \in C(x^*).
\end{aligned}
$$

Equations (41) and (42) indicate that $\nabla_{xx}^2 L(x^*, \lambda)$ is positive semidefinite over $C(x^*)$ while $\Upsilon(x^*, \lambda)$ is positive definite over $C(x^*) \setminus \{0\}$. In this example, the second-order necessary conditions involving the second-order tangent set (42) are stronger than the one not involving the second-order tangent set (41).

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