Metric subregularity

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1 Metric regularity

X, Y—Banach spaces, $F : X \rightrightarrows Y$—a multifunction,
$(\bar{x}, \bar{y}) \in \mathrm{gph}(F) := \{(x, y) \in X \times Y : \ y \in F(x)\}$

Metric regularity of F at $(\bar{x}, \bar{y})$: *there exist* $\kappa, \delta \in (0, \ +\infty)$ *such that*

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta). \qquad (1.1)$$

Strong metric regularity of F at $(\bar{x}, \bar{y})$: *there exist* $\kappa, \delta \in (0, \ +\infty)$ *such that* (1.1) *holds and* $F^{-1}(y) \cap B(\bar{x}, \delta)$ *is a singleton for all* $y \in B(\bar{y}, \delta)$.

1. Strong metric regularity of F at $(\bar{x}, \bar{y}) \Longleftrightarrow F^{-1}$ is a locally Lipschitz single-valued function around $(\bar{x}, \bar{y})$.
2. Metric regularity of F at $(\bar{x}, \bar{y}) \Longleftrightarrow F^{-1}$ is pseudo-Lipschitz around $(\bar{x}, \bar{y})$.
3. Metric regularity of F at $(\bar{x}, \bar{y}) \Longleftrightarrow F$ is locally linear-open around $(\bar{x}, \bar{y})$.

Theorem I (Banach). *Let F be a continuous linear operator between Banach spaces X and Y. Then the following statements are equivalent.*
(i) $F(X) = Y$.
(ii) F is an open mapping.
(iii) F is metrically regular at any point in $\mathrm{gph}(F)$.

Theorem II (Lyusternik-Graves). *Let F be a smooth single-valued function between two Banach spaces such that the derivative $\nabla F(\bar{x})$ is surjective for some $\bar{x}$. Then F is metrically regular at $(\bar{x}, F(\bar{x}))$.*

Theorem III (Robinson-Ursescu). *Let F be a closed convex multifunction between two Banach spaces, and let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ be such that $\bar{y}$ is an interior point of $F(X)$. Then F is metrically subregular at $(\bar{x}, \bar{y})$.*

Coderivatives $\hat{D}^*F(x,y)$, $\bar{D}^*F(x,y)$, $D^*F(x,y) : Y^* \rightrightarrows X^*$

$$\begin{aligned} D^*F(x,y)(y^*) &= \{x^* \in X^* : (x^*, -y^*) \in N(\mathrm{gph}(F), (x,y)\} \\ \bar{D}^*F(x,y)(y^*) &= \{x^* \in X^* : (x^*, -y^*) \in \bar{N}(\mathrm{gph}(F), (x,y)\} \\ \hat{D}^*F(x,y)(y^*) &= \{x^* \in X^* : (x^*, -y^*) \in \hat{N}(\mathrm{gph}(F), (x,y)\} \end{aligned}$$

for all $y^* \in Y^*$.

Theorem IV (Mordukhovich, TAMS 1996). *Let X, Y be finite dimensional spaces and let $F : X \rightrightarrows Y$ be a closed multifunction. Then F is metrically regular at $(\bar{x}, \bar{y})$ if and only if $\bar{D}^*F(\bar{x}, \bar{y})^{-1}(0) = \{0\}$.*

Theorem V(Dontchev et al, TAMS 2004). *Let F be metrically regular at $(\bar{x}, \bar{y})$. Then, there exists $\delta > 0$ such that for any single-valued Lipschitz function $g : X \to Y$ with $\mathrm{lip}(g, \bar{x}) < \delta$, $F + g$ is metrically regular at $(\bar{x}, \bar{y} + g(\bar{x}))$.*

2 Metric subregularity

Metric subregularity of F at $(\bar{x}, \bar{y})$: *there exist* $\tau, \delta \in (0, +\infty)$ *such that*

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \qquad \forall x \in B(\bar{x}, \delta), \tag{2.2}$$

where $F^{-1}(\bar{y})$ can be regarded as the solution set of the following generalized equation

$$\bar{y} \in F(x).$$

In the special case when $F(x) = [\varphi(x), +\infty)$ and $\bar{y} = 0$ (resp. $\bar{y} = \inf_{x \in X} \varphi(x)$), the metric regularity of F at $(\bar{x}, \bar{y})$ reduces to that φ has an error bound (weak sharp minimum) at $\bar{x}$.

Theorem VI (Hoffman, 1952). *Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a polyhedral multifunction (i.e.,$\mathrm{gph}(F)$ is a polyhedron in $\mathbb{R}^m \times \mathbb{R}^n$). Then there exists $\tau > 0$ such that*

$$d(x, F^{-1}(y)) \le \tau d(y, F(x)) \quad \forall (x, y) \in \mathbb{R}^m \times F(\mathbb{R}^m).$$

Theorem VII (Robinson, 1979). *Let $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ be a piecewise polyhedral multifunction (i.e.,$\mathrm{gph}(F)$ is the union of finitely many polyhedra in $\mathbb{R}^m \times \mathbb{R}^n$). Then F is metrically subregular at each $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$.*

Theorem VIII (Zheng-NG, SIOPT, 2014). *Let F be a piecewise polyhedral multifunction between two normed spaces X and Y, and let $\bar{y} \in F(X)$. The following statements hold:*

(1) F is always boundedly metrically subregular at $\bar{y}$, that is, for any $r > 0$ there exists $\tau > 0$ such that

$$d(x, F^{-1}(\bar{y})) \le \tau d(\bar{y}, F(x)) \quad \forall x \in B(0, r).$$

(2) F is globally metrically subregular at $\bar{y}$ if and only if $\lim\limits_{d(x,F^{-1}(\bar{y}))\to\infty} d(\bar{y}, F(x)) = \infty$.

Theorem IX (Ioffe, TAMS, 1984). *Let f be a locally Lipschitz function X and let $\bar{x} \in S(f) = f^{-1}(-\mathbb{R}_+)$. Suppose that there exist $\eta, \delta \in (0, +\infty)$ such that*

$$\eta \leq d(0, \partial f(x)) \quad \forall x \in B(\bar{x}, \delta) \setminus S(f).$$

Then the multifunction $F(x) = [f(x), +\infty)$ is metrically subregular at $(\bar{x}, 0)$.

$$J(y) := \partial\|\cdot\|(y) = \{y^* \in S_{Y^*} | \langle y^*, y\rangle = \|y\|\} \quad \forall y \in Y \setminus \{0\}.$$

For any $\varepsilon > 0$, let

$$J_\varepsilon(y) := \{y^* \in S_{Y^*} | d(y^*, J(y)) < \varepsilon\} \quad \forall y \in Y \setminus \{0\}.$$

For a subset A of Y and $\bar{y} \in Y$,

$$P_A(\bar{y}) := \{y \in A | \|\bar{y} - y\| = d(\bar{y}, A))\}$$

and

$$P_A^\varepsilon(\bar{y}) := \{y \in A | \|y - \bar{y}\| < d(\bar{y}, A) + \varepsilon\}.$$

Theorem 1 *Let F be a closed multifunction between Banach spaces X and Y and let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. The following statements hold:*

(i) Let $\varepsilon, \eta, \delta \in (0,\ +\infty)$ be such that

$$d(0, D_c^* F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

for all $x \in B(a, \delta) \setminus F^{-1}(\bar{y})$ and all $y \in P^{\varepsilon}_{F(x)}(b) \cap B(\bar{y}, \delta)$. Then

$$d(x, F^{-1}(\bar{y})) \leq \frac{1}{\eta} d(\bar{y}, F(x)) \quad \forall x \in B\left(a, \frac{\delta}{2+\eta}\right).$$

(ii) If F is convex, then F is metrically subregular at $(\bar{x}, \bar{y})$ if and only if there exist $\varepsilon \in (0,\ 1)$ and $\eta, \delta \in (0,\ +\infty)$ such that

$$d(0, D_c^* F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

for all $x \in B(a, \delta) \setminus F^{-1}(\bar{y})$ and all $y \in P^{\varepsilon}_{F(x)}(b) \cap B(\bar{y}, \delta)$.

In the case when $F(x) = [f(x),\ +\infty)$, $D_c^* F(x, y)(J_\varepsilon(y - \bar{y})) = \partial_c f(x)$ (resp. $= \emptyset$) for $y = f(x)$ (resp. $y > f(x)$).

For convenience, we adopt $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$ defined by

$$\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta) := \{x \in B(\bar{x}, \delta) \setminus F^{-1}(\bar{y}) : \ d(x, F^{-1}(\bar{y})) > \tau d(\bar{y}, F(x))\}.$$

Clearly, F is metrically subregular at $(\bar{x}, \bar{y})$ with constants τ and δ if and only if $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$ is empty. Therefore, if $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta) \neq \emptyset$ then F is not metrically subregular at $(\bar{x}, \bar{y})$ with the constants τ and δ, but F is possibly metrically subregular at $(\bar{x}, \bar{y})$ with larger constant τ' and smaller constant δ'. This motivates us to establish sufficient conditions for the metric subregularity of F at $(\bar{x}, \bar{y})$ only concerning with x in $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$.

Theorem 2 *Let F be a closed multifunction between Banach spaces X and Y and let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$, and let $\varepsilon, \eta, \delta \in (0, \ +\infty)$ be such that*

$$d(0, D_c^* F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

for all $x \in \mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$ and all $y \in P^{\varepsilon}_{F(x)}(b) \cap B(\bar{y}, \delta)$. Then F is metrically subregular at $(\bar{x}, \bar{y})$.

Adopting an admissible function φ (namely an increasing $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and $[\varphi(t) \to 0 \Rightarrow t \to 0]$), consider the following more general metric subregularity: *F is said to be metrically φ-subregular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ if there exist $\tau, \delta \in (0, +\infty)$ such that*

$$\varphi(d(x, F^{-1}(\bar{y}))) \leq \tau d(\bar{y}, F(x)) \quad \forall x \in B(\bar{x}, \delta).$$

In the special case when $\varphi(t) = t^p$, the metric φ-subregularirty reduces to the so-called Hölder metric subregularity.

For $\varepsilon, \delta, \beta \in (0, +\infty)$, let

$$\mathcal{B}(F, \bar{x}, \bar{y}, \varepsilon, \delta) := \{(x, y) : \; x \in B(\bar{x}, \delta) \setminus F^{-1}(\bar{y}), \; y \in P^{\varepsilon}_{F(x)}(\bar{y}) \cap B(\bar{y}, \delta)\}$$

and

$$K_\beta(\bar{x}, \bar{y}) := \{(x, y) \in X \times Y : \|y - \bar{y}\| \leq \beta \|x - \bar{x}\|\}.$$

Theorem 3 *Let φ be a convex admissible function and F be a closed multifunction between two Banach spaces X and Y. Let $\alpha \in (0,1)$, $\varepsilon, \delta \in (0,+\infty)$, $\beta \in (0, +\infty]$ and $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be such that*

$$\frac{1}{\alpha}\varphi'_+\left(\frac{d(x,F^{-1}(\bar{y}))}{1-\alpha}\right) \leq d(0, D^*F(x,y)(J_\varepsilon(y-\bar{y})))$$

for all $(x,y) \in \mathcal{B}(F,\bar{x},\bar{y},\varepsilon,\delta) \cap K_\beta(\bar{x},\bar{y})$. Let

$$\delta' := \min\left\{\frac{\delta}{1+\alpha}, \varphi^{-1}(\delta)\right\} \quad \text{and} \quad \kappa := \max\left\{1, \frac{\varphi'_+(\delta')}{\alpha\beta}\right\}.$$

Then

$$\varphi(d(x,F^{-1}(\bar{y}))) \leq \kappa d(\bar{y},F(x)) \quad \forall x \in B_X(\bar{x},\delta').$$

3 Convex case

Let $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and C is a closed convex subset of $\mathbb{R}^m$. In the case that

$$F(x) := \big[\max\{f(x), d(x,C)\},\ +\infty\big) \quad \forall x \in \mathbb{R}^m,$$

Lewis and Pang (1997) proved that if $f(\bar{x}) = 0$ and $\partial f(\bar{x}) \neq \emptyset$ then

$$\text{metric subregularity of } F \text{ at } (\bar{x}, 0) \Longrightarrow N(F^{-1}(0), \bar{x}) = \overline{N(C, \bar{x}) + \mathbb{R}_+ \partial f(\bar{x})}$$

Lewis and Pang's open problem: find a useful converse of the abobe implication (characterize the metric regularity via the normal cone identity).

Let φ be a proper lower semicontinuous extended real convex function on a Banach space X and consider the following convex inequality

$$\text{(CIE)} \qquad \varphi(x) \le 0.$$

Let S denote the solution set of (CIE), that is, $S := \{x \in X : \varphi(x) \le 0\}$. In the special case when φ is a continuous convex funciton, recall that (CIE) satisfies basic constraint qualification (BCQ) at $a \in \mathrm{bd}(S)$ if

$$N(S, a) = \mathbb{R}_+ \partial\varphi(a). \tag{3.3}$$

To solve Lewis and Pang's open problem, using the singular subdifferential $\partial^\infty\varphi$, we introduce the following notions:

$$\text{(BCQ)} \qquad N(S, a) = \partial^\infty\varphi(a) + \mathbb{R}_+\partial\varphi(a),$$

$$\text{(SBCQ)} \qquad N(S, a) \cap B_{X^*} \subset \partial^\infty\varphi(a) + [0,\ \tau]\partial\varphi(a).$$

Theorem 4 *Let* φ *be a proper lower semicontinuous convex function on a Banach space* X *and* $F(x) := [\varphi(x),\ +\infty)$ *for all* $x \in X$*. Then* F *is metrically subregular at* $(\bar{x}, 0)$ *with* $\varphi(\bar{x}) = 0$ *if and only if there exist* $\tau, \delta \in (0,\ +\infty)$ *such that the convex function* φ *has strong BCQ at each* $x \in \mathrm{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta)$ *with the same constant* τ*.*

Theorem 5 *Let* F *be a closed convex multifunction between Banach spaces* X *and* Y *and* $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$*. Then* F *is metrically subregular at* $(\bar{x}, \bar{y})$ *if and only if there exists* $\delta > 0$ *such that* F *has strong BCQ at each* $x \in \mathrm{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta)$ *with the same constant, that is, there exists* $\tau \in (0,\ +\infty)$ *such that*

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \tau D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in \mathrm{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta).$$

Theorem 6 *Let F be a closed convex multifunction between Banach spaces X and Y. Then F is globally metrically subregular at $\bar{y} \in F(X)$ (i.e., there exists $\tau \in (0, +\infty)$ such that $d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x))$ for all $x \in X$) if and only if there exist $\tau \in (0, +\infty)$ such that*

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \tau D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in C,$$

where C is some recession core of $F^{-1}(\bar{y})$ in the sense $F^{-1}(\bar{y}) = C + F^{-1}(\bar{y})^{\infty}$. If, in addition, $F^{-1}(\bar{y})$ is a polyhedron, then F is globally metrically subregular at $\bar{y}$ if and only if

$$N(F^{-1}(\bar{y}), x) = D^* F(x, \bar{y})(Y^*) \quad \forall x \in C.$$

Theorem 7 *Let F be a closed multifunction between Banach spaces X and Y and let $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$. Then the following statements hold:*

(i) If F is metrically subregular at $(\bar{x}, \bar{y})$, then there exist $\eta, \delta \in (0, +\infty)$ such that

$$\hat{N}(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \eta D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in F^{-1}(\bar{y}) \cap B(\bar{x}, \delta).$$

(ii) If F is subsmooth at $(\bar{x}, \bar{y})$, F is metrically subregular at $(\bar{x}, \bar{y})$ if and only if there exist $\eta, \delta \in (0, +\infty)$ such that

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \eta D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in F^{-1}(\bar{y}) \cap B(\bar{x}, \delta).$$

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