

# Metric subregularity

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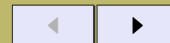
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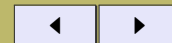
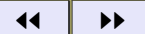
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# 1 Metric regularity

$X, Y$ —Banach spaces,  $F : X \rightrightarrows Y$ —a multifunction,  
 $(\bar{x}, \bar{y}) \in \text{gph}(F) := \{(x, y) \in X \times Y : y \in F(x)\}$

Metric regularity of  $F$  at  $(\bar{x}, \bar{y})$ : there exist  $\kappa, \delta \in (0, +\infty)$  such that

$$d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta). \quad (1.1)$$

Strong metric regularity of  $F$  at  $(\bar{x}, \bar{y})$ : there exist  $\kappa, \delta \in (0, +\infty)$  such that (1.1) holds and  $F^{-1}(y) \cap B(\bar{x}, \delta)$  is a singleton for all  $y \in B(\bar{y}, \delta)$ .

1. Strong metric regularity of  $F$  at  $(\bar{x}, \bar{y}) \iff F^{-1}$  is a locally Lipschitz single-valued function around  $(\bar{x}, \bar{y})$ .
2. Metric regularity of  $F$  at  $(\bar{x}, \bar{y}) \iff F^{-1}$  is pseudo-Lipschitz around  $(\bar{x}, \bar{y})$ .
3. Metric regularity of  $F$  at  $(\bar{x}, \bar{y}) \iff F$  is locally linear-open around  $(\bar{x}, \bar{y})$ .

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**Theorem I (Banach).** *Let  $F$  be a continuous linear operator between Banach spaces  $X$  and  $Y$ . Then the following statements are equivalent.*

- (i)  $F(X) = Y$ .
- (ii)  $F$  is an open mapping.
- (iii)  $F$  is metrically regular at any point in  $\text{gph}(F)$ .

**Theorem II (Lyusternik-Graves).** *Let  $F$  be a smooth single-valued function between two Banach spaces such that the derivative  $\nabla F(\bar{x})$  is surjective for some  $\bar{x}$ . Then  $F$  is metrically regular at  $(\bar{x}, F(\bar{x}))$ .*

**Theorem III (Robinson-Ursescu).** *Let  $F$  be a closed convex multifunction between two Banach spaces, and let  $(\bar{x}, \bar{y}) \in \text{gph}(F)$  be such that  $\bar{y}$  is an interior point of  $F(X)$ . Then  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$ .*

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Coderivatives  $\hat{D}^*F(x, y), \bar{D}^*F(x, y), D^*F(x, y) : Y^* \rightrightarrows X^*$

$$D^*F(x, y)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in N(\text{gph}(F), (x, y))\}$$

$$\bar{D}^*F(x, y)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in \bar{N}(\text{gph}(F), (x, y))\}$$

$$\hat{D}^*F(x, y)(y^*) = \{x^* \in X^* : (x^*, -y^*) \in \hat{N}(\text{gph}(F), (x, y))\}$$

for all  $y^* \in Y^*$ .

**Theorem IV (Mordukhovich, TAMS 1996).** *Let  $X, Y$  be finite dimensional spaces and let  $F : X \rightrightarrows Y$  be a closed multifunction. Then  $F$  is metrically regular at  $(\bar{x}, \bar{y})$  if and only if  $\bar{D}^*F(\bar{x}, \bar{y})^{-1}(0) = \{0\}$ .*

**Theorem V (Dontchev et al, TAMS 2004).** *Let  $F$  be metrically regular at  $(\bar{x}, \bar{y})$ . Then, there exists  $\delta > 0$  such that for any single-valued Lipschitz function  $g : X \rightarrow Y$  with  $\text{lip}(g, \bar{x}) < \delta$ ,  $F + g$  is metrically regular at  $(\bar{x}, \bar{y} + g(\bar{x}))$ .*

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## 2 Metric subregularity

Metric subregularity of  $F$  at  $(\bar{x}, \bar{y})$ : *there exist  $\tau, \delta \in (0, +\infty)$  such that*

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \quad \forall x \in B(\bar{x}, \delta), \quad (2.2)$$

where  $F^{-1}(\bar{y})$  can be regarded as the solution set of the following generalized equation

$$\bar{y} \in F(x).$$

In the special case when  $F(x) = [\varphi(x), +\infty)$  and  $\bar{y} = 0$  (resp.  $\bar{y} = \inf_{x \in X} \varphi(x)$ ), the metric regularity of  $F$  at  $(\bar{x}, \bar{y})$  reduces to that  $\varphi$  has an error bound (weak sharp minimum) at  $\bar{x}$ .

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**Theorem VI (Hoffman, 1952).** Let  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  be a polyhedral multifunction (i.e.,  $\text{gph}(F)$  is a polyhedron in  $\mathbb{R}^m \times \mathbb{R}^n$ ). Then there exists  $\tau > 0$  such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall (x, y) \in \mathbb{R}^m \times F(\mathbb{R}^m).$$

**Theorem VII (Robinson, 1979).** Let  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  be a piecewise polyhedral multifunction (i.e.,  $\text{gph}(F)$  is the union of finitely many polyhedra in  $\mathbb{R}^m \times \mathbb{R}^n$ ). Then  $F$  is metrically subregular at each  $(\bar{x}, \bar{y}) \in \text{gph}(F)$ .

**Theorem VIII (Zheng-NG, SIOPT, 2014).** Let  $F$  be a piecewise polyhedral multifunction between two normed spaces  $X$  and  $Y$ , and let  $\bar{y} \in F(X)$ . The following statements hold:

(1)  $F$  is always boundedly metrically subregular at  $\bar{y}$ , that is, for any  $r > 0$  there exists  $\tau > 0$  such that

$$d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x)) \quad \forall x \in B(0, r).$$

(2)  $F$  is globally metrically subregular at  $\bar{y}$  if and only if

$$\lim_{d(x, F^{-1}(\bar{y})) \rightarrow \infty} d(\bar{y}, F(x)) = \infty.$$

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**Theorem IX (Ioffe, TAMS, 1984).** Let  $f$  be a locally Lipschitz function  $X$  and let  $\bar{x} \in S(f) = f^{-1}(-\mathbb{R}_+)$ . Suppose that there exist  $\eta, \delta \in (0, +\infty)$  such that

$$\eta \leq d(0, \partial f(x)) \quad \forall x \in B(\bar{x}, \delta) \setminus S(f).$$

Then the multifunction  $F(x) = [f(x), +\infty)$  is metrically subregular at  $(\bar{x}, 0)$ .

$$J(y) := \partial \|\cdot\|(y) = \{y^* \in S_{Y^*} \mid \langle y^*, y \rangle = \|y\|\} \quad \forall y \in Y \setminus \{0\}.$$

For any  $\varepsilon > 0$ , let

$$J_\varepsilon(y) := \{y^* \in S_{Y^*} \mid d(y^*, J(y)) < \varepsilon\} \quad \forall y \in Y \setminus \{0\}.$$

For a subset  $A$  of  $Y$  and  $\bar{y} \in Y$ ,

$$P_A(\bar{y}) := \{y \in A \mid \|\bar{y} - y\| = d(\bar{y}, A)\}$$

and

$$P_A^\varepsilon(\bar{y}) := \{y \in A \mid \|y - \bar{y}\| < d(\bar{y}, A) + \varepsilon\}.$$



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**Theorem 1** Let  $F$  be a closed multifunction between Banach spaces  $X$  and  $Y$  and let  $(\bar{x}, \bar{y}) \in \text{gph}(F)$ . The following statements hold:

(i) Let  $\varepsilon, \eta, \delta \in (0, +\infty)$  be such that

$$d(0, D_c^*F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

for all  $x \in B(a, \delta) \setminus F^{-1}(\bar{y})$  and all  $y \in P_{F(x)}^\varepsilon(b) \cap B(\bar{y}, \delta)$ . Then

$$d(x, F^{-1}(\bar{y})) \leq \frac{1}{\eta} d(\bar{y}, F(x)) \quad \forall x \in B\left(a, \frac{\delta}{2 + \eta}\right).$$

(ii) If  $F$  is convex, then  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$  if and only if there exist  $\varepsilon \in (0, 1)$  and  $\eta, \delta \in (0, +\infty)$  such that

$$d(0, D_c^*F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

for all  $x \in B(a, \delta) \setminus F^{-1}(\bar{y})$  and all  $y \in P_{F(x)}^\varepsilon(b) \cap B(\bar{y}, \delta)$ .

In the case when  $F(x) = [f(x), +\infty)$ ,  $D_c^*F(x, y)(J_\varepsilon(y - \bar{y})) = \partial_c f(x)$  (resp.  $= \emptyset$ ) for  $y = f(x)$  (resp.  $y > f(x)$ ).

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For convenience, we adopt  $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$  defined by

$$\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta) := \{x \in B(\bar{x}, \delta) \setminus F^{-1}(\bar{y}) : d(x, F^{-1}(\bar{y})) > \tau d(\bar{y}, F(x))\}.$$

Clearly,  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$  with constants  $\tau$  and  $\delta$  if and only if  $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$  is empty. Therefore, if  $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta) \neq \emptyset$  then  $F$  is not metrically subregular at  $(\bar{x}, \bar{y})$  with the constants  $\tau$  and  $\delta$ , but  $F$  is possibly metrically subregular at  $(\bar{x}, \bar{y})$  with larger constant  $\tau'$  and smaller constant  $\delta'$ . This motivates us to establish sufficient conditions for the metric subregularity of  $F$  at  $(\bar{x}, \bar{y})$  only concerning with  $x$  in  $\mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$ .

**Theorem 2** *Let  $F$  be a closed multifunction between Banach spaces  $X$  and  $Y$  and let  $(\bar{x}, \bar{y}) \in \text{gph}(F)$ , and let  $\varepsilon, \eta, \delta \in (0, +\infty)$  be such that*

$$d(0, D_c^*F(x, y)(J_\varepsilon(y - \bar{y}))) \geq \eta$$

*for all  $x \in \mathcal{N}(F, \bar{x}, \bar{y}, \tau, \delta)$  and all  $y \in P_{F(x)}^\varepsilon(b) \cap B(\bar{y}, \delta)$ . Then  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$ .*



Adopting an admissible function  $\varphi$  (namely an increasing  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\varphi(0) = 0$  and  $[\varphi(t) \rightarrow 0 \Rightarrow t \rightarrow 0]$ ), consider the following more general metric subregularity:  $F$  is said to be metrically  $\varphi$ -subregular at  $(\bar{x}, \bar{y}) \in \text{gph}(F)$  if there exist  $\tau, \delta \in (0, +\infty)$  such that

$$\varphi(d(x, F^{-1}(\bar{y}))) \leq \tau d(\bar{y}, F(x)) \quad \forall x \in B(\bar{x}, \delta).$$

In the special case when  $\varphi(t) = t^p$ , the metric  $\varphi$ -subregularity reduces to the so-called Hölder metric subregularity.

For  $\varepsilon, \delta, \beta \in (0, +\infty)$ , let

$$\mathcal{B}(F, \bar{x}, \bar{y}, \varepsilon, \delta) := \{(x, y) : x \in B(\bar{x}, \delta) \setminus F^{-1}(\bar{y}), y \in P_{F(x)}^\varepsilon(\bar{y}) \cap B(\bar{y}, \delta)\}$$

and

$$K_\beta(\bar{x}, \bar{y}) := \{(x, y) \in X \times Y : \|y - \bar{y}\| \leq \beta \|x - \bar{x}\|\}.$$

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**Theorem 3** Let  $\varphi$  be a convex admissible function and  $F$  be a closed multifunction between two Banach spaces  $X$  and  $Y$ . Let  $\alpha \in (0, 1)$ ,  $\varepsilon, \delta \in (0, +\infty)$ ,  $\beta \in (0, +\infty]$  and  $(\bar{x}, \bar{y}) \in \text{gph}(F)$  be such that

$$\frac{1}{\alpha} \varphi'_+ \left( \frac{d(x, F^{-1}(\bar{y}))}{1 - \alpha} \right) \leq d(0, D^*F(x, y)(J_\varepsilon(y - \bar{y})))$$

for all  $(x, y) \in \mathcal{B}(F, \bar{x}, \bar{y}, \varepsilon, \delta) \cap K_\beta(\bar{x}, \bar{y})$ . Let

$$\delta' := \min \left\{ \frac{\delta}{1 + \alpha}, \varphi^{-1}(\delta) \right\} \quad \text{and} \quad \kappa := \max \left\{ 1, \frac{\varphi'_+(\delta')}{\alpha\beta} \right\}.$$

Then

$$\varphi(d(x, F^{-1}(\bar{y}))) \leq \kappa d(\bar{y}, F(x)) \quad \forall x \in B_X(\bar{x}, \delta').$$

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### 3 Convex case

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper lower semicontinuous convex function and  $C$  is a closed convex subset of  $\mathbb{R}^m$ . In the case that

$$F(x) := [\max\{f(x), d(x, C)\}, +\infty) \quad \forall x \in \mathbb{R}^m,$$

Lewis and Pang (1997) proved that if  $f(\bar{x}) = 0$  and  $\partial f(\bar{x}) \neq \emptyset$  then

metric subregularity of  $F$  at  $(\bar{x}, 0) \implies N(F^{-1}(0), \bar{x}) = \overline{N(C, \bar{x}) + \mathbb{R}_+ \partial f(\bar{x})}$

**Lewis and Pang's open problem:** find a useful converse of the above implication (characterize the metric regularity via the normal cone identity).

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Let  $\varphi$  be a proper lower semicontinuous extended real convex function on a Banach space  $X$  and consider the following convex inequality

$$(CIE) \quad \varphi(x) \leq 0.$$

Let  $S$  denote the solution set of (CIE), that is,  $S := \{x \in X : \varphi(x) \leq 0\}$ . In the special case when  $\varphi$  is a continuous convex function, recall that (CIE) satisfies basic constraint qualification (BCQ) at  $a \in \text{bd}(S)$  if

$$N(S, a) = \mathbb{R}_+ \partial \varphi(a). \quad (3.3)$$

To solve Lewis and Pang's open problem, using the singular subdifferential  $\partial^\infty \varphi$ , we introduce the following notions:

$$(BCQ) \quad N(S, a) = \partial^\infty \varphi(a) + \mathbb{R}_+ \partial \varphi(a),$$

$$(SBCQ) \quad N(S, a) \cap B_{X^*} \subset \partial^\infty \varphi(a) + [0, \tau] \partial \varphi(a).$$

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**Theorem 4** Let  $\varphi$  be a proper lower semicontinuous convex function on a Banach space  $X$  and  $F(x) := [\varphi(x), +\infty)$  for all  $x \in X$ . Then  $F$  is metrically subregular at  $(\bar{x}, 0)$  with  $\varphi(\bar{x}) = 0$  if and only if there exist  $\tau, \delta \in (0, +\infty)$  such that the convex function  $\varphi$  has strong BCQ at each  $x \in \text{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta)$  with the same constant  $\tau$ .

**Theorem 5** Let  $F$  be a closed convex multifunction between Banach spaces  $X$  and  $Y$  and  $(\bar{x}, \bar{y}) \in \text{gph}(F)$ . Then  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$  if and only if there exists  $\delta > 0$  such that  $F$  has strong BCQ at each  $x \in \text{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta)$  with the same constant, that is, there exists  $\tau \in (0, +\infty)$  such that

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \tau D^*F(x, \bar{y})(B_{Y^*}) \quad \forall x \in \text{bd}(F^{-1}(\bar{y})) \cap B(\bar{x}, \delta).$$

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**Theorem 6** Let  $F$  be a closed convex multifunction between Banach spaces  $X$  and  $Y$ . Then  $F$  is globally metrically subregular at  $\bar{y} \in F(X)$  (i.e., there exists  $\tau \in (0, +\infty)$  such that  $d(x, F^{-1}(\bar{y})) \leq \tau d(\bar{y}, F(x))$  for all  $x \in X$ ) if and only if there exist  $\tau \in (0, +\infty)$  such that

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \tau D^*F(x, \bar{y})(B_{Y^*}) \quad \forall x \in C,$$

where  $C$  is some recession core of  $F^{-1}(\bar{y})$  in the sense  $F^{-1}(\bar{y}) = C + F^{-1}(\bar{y})^\infty$ . If, in addition,  $F^{-1}(\bar{y})$  is a polyhedron, then  $F$  is globally metrically subregular at  $\bar{y}$  if and only if

$$N(F^{-1}(\bar{y}), x) = D^*F(x, \bar{y})(Y^*) \quad \forall x \in C.$$

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**Theorem 7** Let  $F$  be a closed multifunction between Banach spaces  $X$  and  $Y$  and let  $(\bar{x}, \bar{y}) \in \text{gph}(F)$ . Then the following statements hold:

(i) If  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$ , then there exist  $\eta, \delta \in (0, +\infty)$  such that

$$\hat{N}(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \eta D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in F^{-1}(\bar{y}) \cap B(\bar{x}, \delta).$$

(ii) If  $F$  is subsmooth at  $(\bar{x}, \bar{y})$ ,  $F$  is metrically subregular at  $(\bar{x}, \bar{y})$  if and only if there exist  $\eta, \delta \in (0, +\infty)$  such that

$$N(F^{-1}(\bar{y}), x) \cap B_{X^*} \subset \eta D^* F(x, \bar{y})(B_{Y^*}) \quad \forall x \in F^{-1}(\bar{y}) \cap B(\bar{x}, \delta).$$

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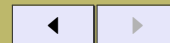
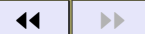
- [1] A. L. Dontchev, A. S. Lewis and R. T. Rockafellar, The radius of metric regularity, Transactions of the American Mathematical Society, 355 (2003), pp.493-517.
- [2] A. L. Dontchev and R. T. Rockafellar, Implicit Functions and Solution Mappings, Springer, New York, 2009.
- [3] A. D. Ioffe, Metric regularity and subdifferential calculus. Russ. Math. Surveys 55 (2000), pp.501-558.
- [4] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation, Springer-Verlag, New York, 2006.
- [5] B. Zhang and X.Y. Zheng, Well-posedness and generalized metric subregularity with respect to an admissible function, Science China Mathematics, 62(2019), pp.809-822.
- [6] X. Y. Zheng and K. F. Ng., Metric regularity and constraint qualifications for convex inequalities in Banach spaces, SIAM J. Optim., 14(2004).pp.757-772.
- [7] X. Y. Zheng and K. F. Ng, Metric Subregularity and Constraint qualifications for Convex Generalized equations in Banach spaces, SIAM J. Optim., 18(2007), pp.437-460.
- [8] X. Y. Zheng and K. F. Ng, Calmness for L-subsmooth multifunctions in Banach spaces, SIAM J. Optim., 19(2009), 1648-1673.
- [9] X. Y. Zheng and K. F. Ng, Metric subregularity and calmness for nonconvex generalized equations in Banach spaces, SIAM J. Optim., 20(2010), pp.2119-2136.



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