BIRATIONAL CLASSIFICATION OF VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract. In this note we survey some classical problems and the recent progresses in birational geomotry in characteristic p.

1. Introduction

We work over an algebraically closed field k. Two varieties X and Y are *birational* means they have two isomorphic open dense subset, algebraically this is equivalent to that the function fields $k(X) \cong k(Y)$ as k-algebras. To classify varieties birationally (modulo birational equivalence), the following the birational invariants are of great significance. Let X be a smooth projective variety of dimension d.

- n-plurigenera: $P_n(X) = h^0(X, nK_X)$ where $K_X \sim \Omega_X^d$ is the canonical divisor.
- Kodaira dimension: $\kappa(X)$ is the stable dimension of $\phi_{nK_X}(X)$ for sufficiently divisible $n > 0$, where ϕ_{nK_X} is the n-canonical map, namely, the map defined by the linear system $|nK_X|$. Note that if for all $n > 0$ $|nK_X| = \emptyset$ then conventionally $\kappa(X) = -\infty$.

If $\kappa(X) \geq 0$, for sufficiently divisible $n > 0$, the map ϕ_{nK_X} gets stable birationally and is called *Iitaka fibration*.

For varieties of fixed dimension d, we can classify varieties according to their Kodaira dimension $\kappa(X) = -\infty, 0, 1, 2, \cdots, d$. To find a reasonably good candidate in a birational class, minimal model program (MMP) was developed. Please refer to [KM98] for basic theory about MMP.

We shall discuss the following topics in characteristic p:

- minimal model theory;
- positivity results and subadditivity of Kodaira dimensions;
- explicit geometry: canonical bundle formula; irregular varieties; varieties with $K_X \equiv 0$; effectivity.

We spend a lot recalling the related results and techniques from characteristic zero for two reasons: to explain the differences happening in positive characteristic and propose reasonable questions.

Conventions:

A *fibration* $f : X \to Y$ means a projective morphism with $f_*\mathcal{O}_X = \mathcal{O}_Y$ (which implies that for $y \in Y$ the fiber X_y is connected).

We use the terminology of minimal model theory from [KM98].

1.1. **Preliminaries.** In this section we recall trace maps of the Frobenius iterations and the notion of F-singularities in positive characteristic. We work over an algebraically closed field in characteristic p.

1.2. **Trace map of the Frobenius map.** For simplicity, first let X be a smooth variety and Δ an effective divisor with index indivisible by p. Assume $(p^g-1)\Delta$ is integral. Denote by $F_X^g : X = X^g \to X$ the g^{th} absolute Frobenius iteration. We have the trace map

$$Tr_{X,\Delta}^{eg} : F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)) \to \mathcal{O}_X$$

which is the composite map of the natural inclusion

$$F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)) \hookrightarrow F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})K_X)$$

and the trace map $Tr_{X_0}^{eg} : F_X^{eg}\mathcal{O}_X((1-p^{eg})K_X) \to \mathcal{O}_X$.

If X is Gorenstein in codimension one (G1) and satisfies Serre condition 2 (S2), we may replace Δ with a rational multiple of an effective divisor and can consider the Frobenius trace map over the Gorenstein part X_0 which can extends to the whole variety since X satisfies S2. If X is normal, without assuming $(p^g-1)\Delta$ is integral, we can modify the trace map $Tr_{X,\Delta}^{eg}$ by replacing the divisor by the integral part $\llcorner(1-p^{eg})(K_X+\Delta)\lrcorner$ to make the map reasonable.

1.3. **Frobenius stable section and direct image.** Now assume X is normal. Let D be an integral divisor on X. Twisting the trace map $Tr_{X,\Delta}^{eg}$ above by $\mathcal{O}_X(D)$ induces a map

$$\begin{aligned} Tr_{X,\Delta}^{eg}(D) :& F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)) \otimes \mathcal{O}_X(D) \\ &\cong F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)+p^{eg}D) \to \mathcal{O}_X(D), \end{aligned}$$

then taking global sections gives

$$H^0(Tr_{X,\Delta}^{eg}(D)) : H^0(X, F_{X*}^{eg}\mathcal{O}_X((1-p^{eg})(K_X+\Delta)+p^{eg}D)) \to H^0(X,D).$$

Let

$$S_\Delta^{eg}(X,D) = \mathrm{Im} H^0(Tr_{X,\Delta}^{eg}(D)) \text{ and } S_\Delta^0(X,D) = \cap_{e\geq 0} S_\Delta^{eg}(X,D).$$

The $S_\Delta^0(X,D) \subseteq H^0(X,D)$ is called the Frobenius stable part and can be attained for sufficiently large e.

Similarly we consider the relative case. Let $f : X \to Y$ be a surjective morphism of normal projective varieties and assume Y is Gorenstein for safety. We may fit the g-th relative Frobenius iteration into the following commutative diagram

$$\begin{tikzcd}
X = X^g \arrow[d, "F^g_{X/Y}"] \arrow[dr, "F^g_X"] \arrow[dd, bend right=60, "f"'] & \\
X_{Y^g} \arrow[r, "\pi^g_Y"] \arrow[d] & X \arrow[d, "f"] \\
Y = Y^g \arrow[r, "F^g_Y"] & Y
\end{tikzcd}$$

here $F^g_{X/Y}: X \to X_{Y^g}$ denote the g^{th} relative Frobenius iteration over Y. By

$$K_{X^{eg}/X_{Y^{eg}}} = (1-p^{eg})K_{X^{eg}/Y^{eg}} \text{ and } F^{eg*}_{X/Y}\pi^{eg*}_Y D = p^{eg}D.$$

we get the trace map

$$Tr^{eg}_{X/Y,\Delta}(D): F^{eg}_{X/Y*}\mathcal{O}_X((1-p^{eg})(K_{X/Y}+\Delta)+p^{eg}D) \to \mathcal{O}_{X_{Y^{eg}}}(\pi^{eg*}_Y D).$$

Applying f_{eg*} to the above map, we get

$$\begin{aligned} f_*Tr^{eg}_{X/Y,\Delta}(D): f_*\mathcal{O}_X((1-p^{eg})(K_{X/Y}+\Delta)+p^{eg}D) \\ \twoheadrightarrow S^{eg}_{\Delta}f_*\mathcal{O}_X(D) \hookrightarrow f_{eg*}\mathcal{O}_{X_{Y^{eg}}}(\pi^{eg*}_Y D) \cong F^{eg*}_Y f_*\mathcal{O}_X(D). \end{aligned}$$

where $S^{eg}_{\Delta}f_*\mathcal{O}_X(D)$ denotes the image of $f_*Tr^{eg}_{X/Y,\Delta}(D)$.

One can prove that for every positive integer e,

$$\dim_{k(\bar\eta)} S^{eg}_{\Delta_{\bar\eta}}(X_{\bar\eta}, D_{\bar\eta}) = \mathrm{rank} S^{eg}_{\Delta}f_*\mathcal{O}_X(D).$$

Consequently for sufficiently large e, $\mathrm{rank}S^{eg}_{\Delta}f_*\mathcal{O}_X(D)$ is stable, which equals to $\dim_{k(\bar\eta)} S^0_{\Delta_{\bar\eta}}(X_{\bar\eta}, D_{\bar\eta})$.

1.4. **F-Singularities.** This notion is defined according to the behavior of the trace map of the Frobenius map. Consider a normal pair (X,Δ). If for any e the trace map

$$Tr^{eg}_{X,\Delta}: F^{eg}_{X*}\mathcal{O}_X(\llcorner(1-p^{eg})(K_X+\Delta)\lrcorner) \to \mathcal{O}_X$$

is surjective then we say (X,Δ) is *F-pure*; and if for any effective divisor D, there exists sufficiently large e such that

$$Tr^{eg}_{X,\Delta}: F^{eg}_{X*}\mathcal{O}_X(\llcorner(1-p^{eg})(K_X+\Delta)-D\lrcorner) \to \mathcal{O}_X$$

is surjective then we say (X,Δ) is *F-regular*. Moreover if the map induced by taking global section of the second trace map is surjective then we say (X,Δ) is globally *F-regular*. If we consider a relative pair $f:(X,\Delta)\to Y$, we can define *relative global F-regularity*.

Here we mention that F-singularities behave very like singularities from minimal model program (see Sec. 2.1), the former ones are defined via Frobenius map while the latter are defined via resolution. The notion F-purity is roughly an analogue of log canonical singularity, and F-regularity is an analogue of klt singularity, global F-regularity is an analogue of log Fano varieties. Note that F-regularity is preserved by small perturbation. Usually the F-notions are stronger, but not preserved in the minimal model program. However, we will see that in lower dimensional cases, the F-notions play an important role in the study of minimal model theory. We refer the reader to [?CT12] for a survey on F-singularities.

2. Minimal model theory

In this section we will summarize the results in minimal model theory in characteristic p. Remind that smooth resolution of singularities in characteristic p is proved only in dimension ≤ 3 ([CP08, CP09]).

2.1. **Singularities in minimal model theory.** To run minimal model program, we have to permit mild singularities to guarantee working in a closed category of varieties. More generally we usually consider a pair (X, Δ) where $\Delta = \sum_i c_i D_i$ is an effective divisor with coefficients $c_i \in [0, 1]$. If for every smooth log resolution $\rho : Y \to (X, \Delta)$, in the formula $K_Y = \rho^*(K_X + \Delta) + \sum_i a_i E_i$, each $a_i > -1$ (≥ -1) then we say (X, Δ) is *Kawamata log terminal* or *klt* for short (*log canonical* or *lc*); if $\Delta = 0$ and for every exceptional divisor E_i the coefficient $a_i > 0$ (≥ 0) we say X is *terminal* (*canonical*).

2.2. **Surfaces.** For surfaces, the minimal model program and abundance for log canonical pairs have been established. Precisely

- the case $\Delta = 0$ was proved by Bombieri and Mumford in 1970s;
- the log case is over an algebraically closed field and over an F-finite field is established by Tanaka [Tan14, Tan16], and over an F-finite field is also established by Tanaka [Tan18a, Tan18b, Tan20].

An advantage of treating surfaces is the Riemann-Roch formula, which provides great convenience to obtain global sections.

2.3. **Minimal model program for threefolds.** Up to this moment, minimal model program has been established for klt pairs when $p \geq 5$. When $p > 5$, Hacon and Xu [HX15] first proved the existence of minimal model for klt pairs (X, Δ) with standard coefficients and $K_X + \Delta$ is psudo-effective; then Birkar [Bir16] treated the nonstandard case and joint with Waldron [BW17] proved the existence of Mori fiber space when $K_X + \Delta$ is not pseudo-effective. Hacon and Witaszek [HW19] recently treat the case $p = 5$.

We explain the novelties to prove MMP for threefolds in characteristic p. We refer the reader to [Kol92] for the proof of MMP in characteristic zero, particularly in the proof Kawamata-Viehweg vanishing theorem plays an important role to lift sections on a closed subvariety, for example this implies the log canonical center is normal. In [HX15], the authors proved existence of pl-flips for pairs with standard coefficients in characteristic $p > 5$. The role of Kawamata-Viehweg vanishing theorem is replaced by a combination of Fujita's vanishing theorem and Mumford regularity, so under suitable conditions, those Frobenius stable sections $S^0(X, K_X + D)$ are liftable. In [HX15], they want to prove $S^0(X, K_X + D) = H^0(X, K_X + D)$, hence they have to assume the coefficients standard, because their argument relies on the result that a relative del-pezzo surface $(S, B) \to T$ with standard coefficients is globally F-regular ([Har98]). Birkar [Bir16] treated the general case by perturbing the coefficient one by one. To treat the case characteristic 5, Hacon and Witaszek [HW19] make new subtle observations and rectifications. For pathologies in characteristic $p = 2, 3$, please refer to [CT19].

2.4. **Abundance for threefolds.** To investigate abundance, we try to adapt the method from characteristic zero to characteristic p, so we separate into two cases according to the irregularity $q(X) = \frac{b_1}{2} = \dim \mathrm{Pic}^0(X)$.

(1) When $p \geq 5$ and $q(X) \geq 1$, abundance has been proved for minimal threefolds by the author [Zha20a], in fact the author only treated the case $p > 5$, but by Hacon and Witaszek's recent work, the proof also applies when $p = 5$. The advantage of

the condition $q(X) \geq 1$ lies in that X is equipped with the Albanese map, then we get a natural fibration by doing Stein factorization

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\mathrm{alb}_X"] & \\ Y \arrow[r, "\mathrm{alb}_Y"] & \mathrm{Alb}(X) \end{tikzcd}.$$

The problem is reduced to proving subadditivity of Kodaira dimension which will be discussed later.

(2) When $q = 0$ and $p \geq 5$, only nonvanishing was proved in [XZ19] by following Miyaoka's strategy. Comparing with characteristic zero case, one need to tackle with the new phenomena: $\rho^* K_X \cdot c_2(Y) < 0$, in this case we have a foliation by rational curves, then show the full abundance under this assumption.

2.5. **Open problems.** About the minimal model theory for threefold in characteristic p, the following problems are still open.

Question 1. Abundance for minimal models when $p \geq 5$.

As MMP has been established, this is almost accessible by adapting the strategy from characteristic zero. As we have nonvanishing, the main difficulty is how to extend a pluricanonical section from a closed divisor to the whole variety.

Question 2. Minimal model program when $p = 2, 3$.

Comparing with the case $p \geq 5$, to prove the existence of flip, when lifting sections on a plt center, one need this surface F-regular under this flipping contraction, but this condition fails when $p < 5$. Please refer to [HX15, HW19] for the subtleness.

Question 3. Nonvanishing and abundance for minimal models when $p = 2, 3$.

Comparing with the case $p \geq 5$, the main difficulty is to treat quasi-elliptic fibration, which happens only in characteristic $p = 2, 3$. In fact we are in lack of a reasonable canonical bundle formula, which will be discussed later.

3. Positivity

Positivity is closely related to constructing global sections, in particular is essential in the study of abundance. We will focus on the the direct image of relative pluricanonical sheaves and compare the techniques in characteristic p and zero.

3.1. **Several positivities.** We introduce several positivities weaker than ampleness.

Definition 3.1. A locally free coherent sheaf $\mathcal{F}$ on a normal quasi-projective variety Y is said to be *nef* if for every surjective homomorphism $\mathcal{F} \to \mathcal{G}$, $\det \mathcal{G}$ is nef.

Definition 3.2. A torsion free coherent sheaf $\mathcal{F}$ on a normal quasi-projective variety Y is said to be *weakly positive*, if for every ample line bundle H on Y and positive integer m, there exists a sufficiently large integer n such that, $S^n(H \otimes S^m(\mathcal{F})^{**})$ is generically globally generated, where for a coherent sheaf $\mathcal{G}$, $\mathcal{G}^{**} := \mathcal{H}om(\mathcal{H}om(\mathcal{G}, \mathcal{O}_Y), \mathcal{O}_Y)$ denotes the double dual.

Remark 3.3. (1) Weak positivity is introduced by Viehweg [Vie83] to study subadditivity of Kodaira dimensions.

(2) Nefness implies weak positivity, but when Y is a curve, they are equivalent.

3.2. **Results in characteristic zero.** We only concern the positivity of the push-forward of the relative pluricanonical sheaves. There are so many related results, here we only recall some classical ones that are applied widely in birational geometry.

Theorem 3.4. *Let $f : X \to Y$ be a fibration of nonsingular projective varieties. Then for any $n > 0$, the sheaf $f_*\omega^n_{X/Y}$ is weakly positive.*

Ingredient of the proof: The case $\dim Y = 1$ was first proved by [Fuj78] $\dim Y = 1$, and by [Kaw83] for any $\dim Y$. The proof follows from applying VHS (variation of hodge structure) and local calculating the curvature of $f_*\omega_{X/Y}$ (after certain base change), which is equipped with a hermitian metric induced by the natural pairing. For general n, it was proved by [Vie83] by applying the above result to the fiber product

$$f^r : X \times_Y X \times_Y \cdots_Y X \to Y.$$

Furthermore, if the period map of the family $f : X \to Y$ is locally injective at the general point then the $f_*\omega^n_{X/Y}$ is big for $n \geq 2$.

These results are generalized to log pairs by VMHS (variation of mixed hodge structure) in [FF14].

Theorem 3.5. *Let $f : (X, \Delta) \to Y$ be a fibration from a G_1, S_2 pair to a curve. Assume (X_y, Δ_y) is semi-log canonical for general y, and $k(K_{X/Y} + \Delta)$ is Catier. Then $f_*\mathcal{O}_X(k(K_{X/Y} + \Delta)$ is weakly positive.*

Remark 3.6. Weak positivity is very important in birational geometry, for example it is crucial

- to prove the projectivity of moduli spaces,
- to find sections, say, prove Iitaka conjecture.

3.3. **Results in characteristic p.** The analogous result is not true in characteristic $p > 0$. There is an example $f : X \to Y$ of semi-stable fibration of a surface such that $f_*\omega_{X/Y}$ is not nef, and Raynaud's example is a fibration $f : X \to Y$ of surface with $X_{\bar{\eta}}$ being singular, the sheaves $f_*\omega^n_{X/Y}$ is negative for any n.

Patakfalvi [Pat14] first gets a positivity for fibrations over curves under assumption that K_X is relatively ample and fibers are smooth (or mildly singular).

Theorem 3.7. *Let $f : X \to Y$ be a surjective morphism from a normal projective variety to a smooth curve. Let Δ be an effective $\mathbb{Q}$-divisor on X such that $p \nmid \mathrm{ind}(K_X + \Delta)$. Assume that for general $y \in Y$, the fiber (X_y, Δ_y) is sharply F-pure.*

(i) If $K_X + \Delta$ is f-ample, then $f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ is nef.

(ii) If $K_X + \Delta$ is f-nef, then $K_{X/Y} + \Delta$ is nef.

A natural question is how to choose m satisfying (i) above. Shortly later, [Eji17] generalizes above Patakfalvi's result and tells how to determine such m.

Theorem 3.8. *Let $f : X \to Y$ be a separable surjective morphism between smooth projective varieties, let Δ be an effective $\mathbb{Q}$-divisor on X, and let a be a positive integer prime to p such that $a\Delta$ is integral. Assume that*

(i) the $k(\bar\eta)$-algebra $\bigoplus_{m\geq 0} H^0(X_{\bar\eta}, m(a(K_{X/Y}+\Delta)_{\bar\eta}))$ is generated by $H^0(X_{\bar\eta}, a(K_{X/Y}+\Delta)_{\bar\eta})$; and

(ii) $S^0_{\Delta_{\bar\eta}}(X_{\bar\eta}, a(K_{X/Y}+\Delta)_{\bar\eta})) = H^0(X_{\bar\eta}, a(K_{X/Y}+\Delta)_{\bar\eta}))$.

Then $f_*\mathcal{O}_X(a(K_{X/Y}+\Delta))$ is weakly positive.

Remark 3.9. When $K_{X_{\bar\eta}}+\Delta_{\bar\eta}$ is ample and $(X_{\bar\eta}, \Delta_{\bar\eta})$ is F-pure, then (i, ii) holds for sufficiently divisible a. So Ejiri's result implies Patakfalvi's.

Patakfalvi's proof is a combination of trace maps of absolute Frobenius iterations and Viehweg's fiber product trick, while Ejiri mainly applies trace maps of relative Frobenius maps and gets positivity by an iteration of sheaves. We will explain Ejiri's proof later.

In fact Ejiri proves the sheaf in Theorem 3.8 satisfies a stronger positivity.

Definition 3.10. Let Y be a quasi-projective variety, $\mathcal{F}$ a torsion free coherent sheaf and H an ample $\mathbb{Q}$-Cartier divisor on Y. Let

$$t(Y,\mathcal{F},H) = \sup\{t\in\mathbb{Q} | \text{the sheaf } (F_Y^{e*}\mathcal{F})\otimes\mathcal{O}_Y([-p^etH]) \text{ is generically globally generated for some } e>0\}.$$

We say $\mathcal{F}$ is *FWP (Frobenius weakly positive)* if for an ample $\mathbb{Q}$-Cartier divisor H, $t(Y,\mathcal{F},H)\geq 0$, equivalently, there exist a sequence of positive integers $\{n_e | e = 1,2,3,\cdots\}$ such that n_eH is Cartier, the sheaf $(F_Y^{e*}\mathcal{F})\otimes\mathcal{O}_Y(n_eH)$ is generically globally generated and $\frac{n_e}{p^e}\to 0$ as $e\to+\infty$.

This property is independent of the choice of the ample divisor H.

Unfortunately when $(X_{\bar\eta}, \Delta_{\bar\eta})$ is wildly singular the condition (ii) of Theorem 3.8 does not hold. If granted some numerical condition we can treat this bad case ([PSZ18] and [Zha19a]), so we have the following result.

Theorem 3.11. *Let X be a normal projective variety and Y a smooth projective variety over an algebraically closed field k with* char $k = p > 0$. *Let $f : X \to Y$ be a separable surjective projective morphism. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_{X/Y}+\Delta$ is $\mathbb{Q}$-Cartier and $p\nmid \operatorname{ind}(K_{X/Y}+\Delta)$. If D is a Cartier divisor on X such that $D-K_{X/Y}-\Delta$ is nef and f-semi-ample, then for sufficiently divisible g, the sheaf $F_Y^{g*}f_*\mathcal{O}_X(D)$ contains a FWP sub-sheaf $S^g_\Delta f_*\mathcal{O}_X(D)$ of rank $\dim_{k(\bar\eta)} S^0_{\Delta_{\bar\eta}}(X_{\bar\eta}, D_{\bar\eta})$.*

Remark 3.12. In fact if D is relatively big, we have that for sufficiently divisible n $\dim_{k(\bar\eta)} S^0_{\Delta_{\bar\eta}}(X_{\bar\eta}, nD_{\bar\eta}) > 0$ ([Zha19a, Prop. 2.5]).

If granted MMP, the above result, in the study of subadditivity of Kodaira dimension, can substitute the role of the weak positivity of $f_*\omega^n_{X/Y}$ in characteristic zero. For example, if $K_X+\Delta$ is nef, relatively big and $p\nmid\operatorname{ind}(K_{X/Y}+\Delta)_\eta$, then for sufficiently large a, $F_Y^{g*}(f_*\mathcal{O}_X(a(K_{X/Y}+\Delta))\otimes\omega_Y^{a-1})$ contains a nonzero FWP subsheaf.

3.4. **The approach to positivity in characteristic p.** We will introduce the notion of Frobenius stable sections and sketch the proof of Theorem 3.8 and 3.11, which are so typical.

Before preceding with the proof, I outline the strategy. FWP is defined by global generation, this is usually obtained by Mumford regularity theorem, so we need certain vanishing theorems. In positive characteristic, the only vanishing theorem is Fujita's vanishing theorem ([Kee03]), which is a generalization of Serre vanishing theorem.

Lemma 3.13. *(***Relative Fujita Vanishing***) Let $f : X \to Y$ be a projective morphism over a Noetherian scheme, H an f-ample line bundle and $\mathcal{F}$ a coherent sheaf on X. Then there exists a positive integer N such that, for every $n > N$ and every nef line bundle L*

$$R^i f_*(\mathcal{F} \otimes H^n \otimes L) = 0, \text{ if } i > 0.$$

To apply this vanishing, we need to do Frobenius amplitude by applying the trace map of the (relative) Frobenius map, this is why these positive results hold for Frobenius stable sections.

To prove Theorem 3.8, we set $D = a(K_{X/Y} + \Delta)$. Then the condition (ii) implies that

$$\begin{aligned} f_*Tr^{eg}_{X/Y,\Delta}(a(K_{X/Y} + \Delta)) : f_*\mathcal{O}_X((1 - p^{eg} + ap^{eg})(K_{X/Y} + \Delta)) \\ \to F_Y^{eg*} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)). \end{aligned}$$

is generically surjective. Let H be an ample divisor on Y. Let $t_0 = t(Y, f_*\mathcal{O}_X(a(K_{X/Y}+\Delta)), H)$.

(1) Only need to show that $t_0 \geq 0$.

(2) By definition,

$$t(Y, S^k f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)), H) \geq kt_0$$

and

$$t(Y, F_Y^{g*} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)), H) = p^g t_0.$$

(3) By (ii), the trace map of relative Frobenius

$$\begin{aligned} Tr_{X/Y} : &F^g_{X/Y*}\mathcal{O}_X((1 - p^g)(K_{X/Y} + \Delta) + p^g a(K_{X/Y} + \Delta)) \\ &\cong F^g_{X/Y*}\mathcal{O}_X((1 - p^g + ap^g)(K_{X/Y} + \Delta)) \to F_Y^{g*} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)). \end{aligned}$$

is surjective over the generic point of Y^g.

(4) By (i), if denoting $k_g = \frac{p^g-1}{a}$ then

$$\begin{aligned} S^{p^g-k_g} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)) \to F^g_{X/Y*}\mathcal{O}_X((1 - p^g + ap^g)(K_{X/Y} + \Delta)) \\ \to F_Y^{g*} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)) \end{aligned}$$

is surjective over the generic point of Y^g.

(5) We conclude from the above surjection that

$$t(Y, S^{p^g-k_g} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)), H) \leq t(Y, F_Y^{g*} f_*\mathcal{O}_X(a(K_{X/Y} + \Delta)), H)$$

hence

$$(p^g - k_g)t_0 \leq p^g t_0$$

which implies the theorem.

Proof of Theorem 3.11. We can take g is divisible enough that for every positive integer e, the sheaf $S^{eg}_{\Delta} f_*\mathcal{O}_X(D)$ has the stable rank $\dim_{k(\bar\eta)} S^0_{\Delta_{\bar\eta}}(X_{\bar\eta}, D_{\bar\eta})$. Then for every integer $e > 0$, the composite homomorphism below is generically surjective

$$\begin{aligned}\alpha^{eg} : f_*\mathcal{O}_X((1-p^{eg})(K_{X/Y}) - [(p^{eg}-1)\Delta] + p^{eg}D) \\ \twoheadrightarrow (S^{eg}_{\Delta} f_*\mathcal{O}_X(D)) \hookrightarrow (F_Y^{(e-1)g*} S^g_{\Delta} f_*\mathcal{O}_X(D)),\end{aligned}$$

because the two sheaves $S^{eg}_{\Delta} f_*\mathcal{O}_X(D)$ and $(F_Y^{(e-1)g*} S^g_{\Delta} f_*\mathcal{O}_X(D))$ have the same rank.

Let H be an ample Cartier divisor on Y. Tensoring the map α^{eg} with $\mathcal{O}_Y(eH)$, we get a generically surjective homomorphism

$$\begin{aligned}\beta^{eg} : f_*\mathcal{O}_X((1-p^{eg})(K_{X/Y}) - [(p^{eg}-1)\Delta] + p^{eg}D + ef^*H) \\ \to F_Y^{(e-1)g*} S^g_{\Delta} f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(eH).\end{aligned}$$

From now on for simplicity, we only consider the case $D - (K_{X/Y} + \Delta)$ is **nef and f-ample**. Then we find that the divisor

$$(1-p^{eg})(K_{X/Y}) + \Delta) + p^{eg}D + ef^*H = (p^{eg}-1)(D - (K_{X/Y}) + \Delta)) + ef^*H$$

is very very ample as e is big, hence the sheaf

$$f_*\mathcal{O}_X((1-p^{eg})(K_{X/Y}) + \Delta) + p^{eg}D + ef^*H)$$

is generically globally generated. (Dtails are left as an exercise)

This implies that the image of β^{eg} is generically globally generated, hence so is the sheaf $F_Y^{(e-1)g*} S^g_{\Delta} f_*\mathcal{O}_X(D) \otimes \mathcal{O}_Y(eH)$. Therefore, the sheaf $S^g_{\Delta} f_*\mathcal{O}_X(D)$ is FWP.

3.5. **Questions and comments.**

Question 4. Let $f : X \to Y$ be a fibration to a curve. Assume the generic fiber is smooth and K_X is relatively semiample. We expect that for sufficiently divisible $m > 0$, the sheaf $f_*\omega^m_{X/Y}$ is nef.

By [Pat14], $K_{X/Y}$ is nef, so if $\kappa(F) = 0$, then the above statement is true.

4. Subadditivity of Kodaira dimension

It is a central problem to construct pluricanonical sections in birational geometry. In characteristic zero Iitaka proposed

Conjecture 4.1 (Iitaka's conjecture)**.** *Let $f : X \to Y$ be a fibration between smooth projective varieties over an algebraically closed field k, with $\dim X = n$ and $\dim Y = m$. Then the Kodaira dimension satisfies subadditivity*

$$C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar\eta}, K_{X_{\bar\eta}}).$$

Kawamata, Kollár, Viehweg did pioneering work in 1980s, and Chen-Hacon, Cao-Paun etc. also made remarkable contributions to this conjecture. Up to now $C_{n,m}$ has been proved

(1) when F has a good minimal model by [Kaw85],
(2) when $\dim Y = 1$ by [Kaw82] and when $\dim Y = 2$ by [Cao18],
(3) when Y is of maximal Albanese dimension by [CP17, HPS18].

It is worthwhile mentioning that the Albanese map $a_X : X \to A$ induces a natural fibration $f : X \to Y$ by Stein factorization, where Y is of maximal Albanese dimension, so this case is of special interest. In dimension ≤ 3, subadditivity almost implies abundance for the case $q(X) > 0$, which was proved by Viehweg.

4.1. **The formulation in characteristic p.** In characteristic $p > 0$, it is interesting whether subadditivity of Kodaira dimension holds. Remind that the geometric fiber $X_{\bar{\eta}}$ is often singular (not even reduced if f is inseparable). We may expect

$$WC_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(\tilde{X}_{\bar{\eta}})$$

where $\tilde{X}_{\bar{\eta}}$ is a smooth model of the scheme with reduced structure of $X_{\bar{\eta}}$. But this may be too weak, because it is far from inducing abundance in low dimensional cases. As X is assumed smooth, the dualizing sheaf $\omega_{X_{\bar{\eta}}}$ is invertible, hence the Kodaira dimension $\kappa(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}})$ is well defined. We may expect

$$C_{n,m} : \kappa(X) \geq \kappa(Y) + \kappa(X_{\bar{\eta}}, K_{X_{\bar{\eta}}}).$$

Remark that the first inequality $WC_{n,m}$ does be weaker than the second one, because if $\omega_{X_{\bar{\eta}}}$ has a smooth birational model $\omega_{\tilde{X}_{\bar{\eta}}}$ then always

$$\kappa(X_{\bar{\eta}}, \omega_{X_{\bar{\eta}}}) \geq \kappa(\tilde{X}_{\bar{\eta}}).$$

Up to now in characteristic p the following are known

(1) $WC_{n,n-1}$ and $WC_{3,1}$ ([CZ15, EZ18]);
(2) $C_{3,1}$ when Y is of m.A.d. and $p > 5$, which implies abundance for the case $q(X) > 0$ ([Zha20a]);
(3) the fiber product of Raynaud surfaces fails $C_{n,m}$ ([CEKZ20]).

4.2. **The idea to prove subadditivity of Kodaira dimension.** We explain and compare the differences in characteristic zero and $p > 0$ as follows.

(1) **Positivity plays the central role.** In characteristic zero or with restrictive condition on the fibers in chacteristic p, we have weak positivity of $\mathcal{F} = f_*\omega_{X/Y}^n$, and assuming minimal model theory we also have a positive subsheaf of $\mathcal{F} \subseteq F_Y^{e*}(f_*\omega_{X/Y}^n \otimes \omega_Y^{n-1})$. When Y is of general type or $\det \mathcal{F}$ is big, it is easy to conclude subadditivity. Comparing with WP, FWP is easier to lead to subadditivity ([Zha19a, Theorem 4.1]).

(2) **How to get sections of $\mathcal{F}$ up to torsion.** We can often reduce to treat the extremal cases $\kappa(Y) = 0$ and $\det \mathcal{F} = 0$, for example, when Y is of m. A. d., by doing induction we can reduce to the case Y is an abelian variety ([HPS18]). Let's focus on the case that Y is an abelian variety.

- In characteristic zero, when Y is an elliptic curve, $\mathcal{F}$ splits into direct sum of stable vector bundles with numerically trivial determinants, these determinants are torsion thanks to Simpson's results in Hodge theory ([Kaw82]). When Y is an abelian variety, Cao and Păun [CP17] can endow $\mathcal{F}$ with a good metric, hence it is from the representation of $\pi_1(Y)$, and in turn they split $\mathcal{F}$ into line bundles since $\pi_1 Y = \oplus\mathbb{Z}$.

- In characteristic p, the first method to get some torsion property is using trace maps of relative Frobenius maps ([Eji17, EZ18]), which can treat the case when Y is an elliptic curve and fibers are smooth. In general, it is much more difficult, by combining Fourier-Mukai transform and taking advantage of trace map of the Frobenius iterations and "killing cohomology", [Zha20a] can prove that after some base change $\pi : A' \to A$ of abelian varieties, there is a surjective map
$$\oplus_i P_i \to \pi^* \mathcal{F} \text{ where } P_i \in \mathrm{Pic}^0(A'),$$
then using adjunction formula, abundance for surfaces and techniques of MMP, when Y is of m.A.d., Iitaka conjecture is proved in dimension 3 and char $p > 5$.

4.3. **Further questions and comments.** The argument of [Zha20a] highly relies on the minimal model theory. For the reason why $C_{3,m}$ holds in positive characteristic when Y is of m.A.d., we attribute to the relative minimal model over Y is in fact minimal, hence we have the positivity. In fact if granted Cone Theorem, it is enough to assume Y is non-uniruled, so in this case we [CEKZ20] expect subadditivity. If the fibers are smooth, we also expect positivity results and subadditivity.

5. Effectivity of pluricanonical systems

By effectivity of pluricanonical systems we concerns

Question 5. For varieties with $\kappa(X) \geq 0$, how to find numbers N_1, N_2 as small as possible such that

- $|N_1 K_X| \neq \emptyset$ (effective nonvanishing)
- effective Iitaka fibration: $|N_2 K_X|$ defines a rational map birationally equivalent to the Iitaka fibration (effective Iitaka fibration problem).

These are classical problems in birational geometry.

5.1. **Results and the strategies in characteristic zero.** First we consider the case the case that K_X is big and collect several classical strategies frequently utilized in the literature. More generally these strategies can treat adjoint linear system $K_X + L$ where L is big.

- $\dim(X) = 2$: Reider's method ([Rei88];
- $\dim(X) \leq 3$: a combination of Riemann-Roch formula and dimension reduction ([CC10a, CC10b]);
- arbitrary dimensional case: a combination of Kawamata-Viehweg vanishing and cutting isolated log canonical center ([HM06, Tak06])

For a smooth projective surface X of general type over an algebraically closed field of arbitrary characteristic, it is known that $|5K_X|$ is birational ([Bom73, Rei88]). In general, over the field of complex numbers $\mathbb{C}$, there exists a number $M(d)$ such that, for any d-dimensional smooth projective varieties of general type, if $m \geq M(d)$ then $|mK_X|$ is birational ([HM06, Tak06]), and for threefolds we may take $M(3) = 126$ ([CC10a, CC10b]).

For varieties with intermediate Kodaira dimension $0 \leq \kappa(X) < \dim X$, the following are known

- When $\dim X = 2$, $|12K| \neq \emptyset$ which was known due to Castelnuovo and Enriques more than 100 years ago, and $|mK|$ defines Iitaka fibration for $m \geq 96$ and divisible by 12 [Iit72].
- When $\dim X = 3$ and $\kappa(X) = 0$, there is a computable number $n < 1000$ such that $nK_X \sim 0$ [Kaw86].
- In general, there exists a number $m(d, b_F, \beta_F)$ where $|b_F K_F| \neq 0$, $\beta_F = h^{d_F}(\tilde{F})$ ($\tilde{F}$ is the canonical cover of F), such that for N divisible by m, $|NK|$ defines a map birational to Iitaka fibration [BZ16].

5.2. **Results in characteristic p.** For curves and surfaces in char p, the classical results are the same as the case in characteristic zero, non-general type case is due to Bombieri-Mumford, refer to the book [Băd01]; the general type case is due to [Eke88]. The key ingredients include

- Riemann-Roch formula;
- Result from topology, say, Noether's formula;
- Classical geometric methods, for example, intersection of divisors, linear system and dimension reduction;
- Reider's method.

Remind that we should not expect the same result for the adjoint linear system. For example, Fujita conjecture predicts that for a surface X and an ample line bundle L, $K_X + 3L$ is base point free. This has been proved for surfaces in char 0 by Reider's method. However, in char p for any large n there exist a generalized Raynaud's surfaces and ample line bundle L such that $K + nL$ is not base point free, see [GZZ20].

5.3. **The inductive approaches in characteristic p.** We briefly recall a classical inductive strategy from characteristic zero as follows. For a smooth projective variety X over an algebraically closed field of characteristic zero, if given a natural number n_1 such that $\dim |n_1 K_X| \geq 1$, which induces a rational map $f : X \dashrightarrow Y$ with generic fiber F, and given a number n_2 such that $|n_2 K_F|$ defines birational map of F, then one can get a suitably bigger number $M(n_1, n_2)$ such that for $m \geq M(n_1, n_2)$ the linear system $|mK_X|$ is birational. The most important step to carry out this strategy is to extend sections on a fiber to the whole variety, hence one needs vanishing results and weak positivity of the pushforward of (relative) pluricanonical sheaves. To adapt this approach to characteristic p, the author [Zha20b] applies a more strict Frobenius stable sections defined as follows. To treat a big divisor D, we need to disturb D to be ample. First fix $\Delta \geq 0$ and let

$$S^e_\Delta(X, K_X + D) := S^e(X, K_X + D - \Delta) \otimes s_E \subseteq S^e(X, K_X + D)$$

where $E = \ulcorner D \urcorner - \ulcorner D - \Delta \urcorner$, and respectively

$$S^0_\Delta(X, K_X + D) = \bigcap_{e \geq 0} S^e_\Delta(X, K_X + D).$$

Next assume D is nef and big. Let Θ_D^{amp} denote the set of effective divisors Δ such that $D - \Delta$ is ample and define

$$S_-^0(X, K_X + D) = \bigcap_{\Delta \in \Theta_D^{\mathrm{amp}}} \big(\bigcup_{t \in \mathbb{Q}^+} S_{t\Delta}^0(X, K_X + D) \ \subseteq S^0(X, K_X + D)\big).$$

To do induction, we have the following two theorems.

Theorem 5.1. *Let $f : X \to Y$ be a fibration of normal projective varieties over an algebraically closed field k of characteristic p, and let $d = \dim Y$. Let D be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and $H, \tilde{H}$ two $\mathbb{Q}$-Cartier Weil divisors on Y such that $|H|$ defines a generically finite map and $|\tilde{H}|$ is birational.*

(i) If $S_-^0(X_\eta, K_{X_\eta} + D|_{X_\eta}) \neq 0$ then $S_-^0(X, K_X + D + f^*sH) \neq 0$ for any $s \geq d$.

(ii) If $S_-^0(X_\eta, K_{X_\eta} + D|_{X_\eta})$ is birational then $S_-^0(X, K_X + D + f^*sH)$ is birational for $s \geq d+1$; and if moreover $S_-^0(X, K_X + D + df^*H - f^*\tilde{H}) \neq 0$ then $S_-^0(X, K_X + D + f^*dH)$ is birational.

Next criterion is inspired by the idea of continuous global generation (CGG) introduced by Pareschi and Popa [PP03], which is used to treat the case of irregular varieties.

Theorem 5.2. *Let X be a smooth projective variety over an algebraically closed field k of characteristic p, and let $a : X \to A$ be a morphism to an abelian variety. Denote by $f : X \to Y$ the fibration arising from the Stein factorization of $a : X \to A$. Let D, D_1, D_2 be three divisors on X. Assume that D is nef, big and $\mathbb{Q}$-Cartier.*

(i) If $S_-^0(X_\eta, K_{X_\eta} + D_\eta) \neq 0$, then for any $\mathcal{P}_\alpha \in \mathrm{Pic}^0(A)$, $H^0(X, K_X + \ulcorner D \urcorner + a^*\mathcal{P}_\alpha)) \neq 0$, and there exists some $\mathcal{P}_\beta \in \mathrm{Pic}^0(A)$ such that $S_-^0(X, K_X + \ulcorner D \urcorner + a^*\mathcal{P}_\beta) \neq 0$.

(ii) Assume that $S_-^0(X_\eta, K_{X_\eta} + D_\eta) \neq 0$, D_1 is integral and for any $\mathcal{P}_\alpha \in \mathrm{Pic}^0(A)$, $|D_1 + a^*\mathcal{P}_\alpha| \neq \emptyset$. Then for any $\mathcal{P}_{\alpha_0} \in \mathrm{Pic}^0(A)$, $S_-^0(X, K_X + D + D_1 + a^*\mathcal{P}_{\alpha_0}) \neq 0$.

(iii) Assume that $S_-^0(X_\eta, K_{X_\eta} + D_\eta)$ is birational, both D_1 and D_2 are integral and for any $\mathcal{P}_\alpha \in \mathrm{Pic}^0(A)$, $|D_i + a^*\mathcal{P}_\alpha| \neq \emptyset$. Then for any $\mathcal{P}_{\alpha_0} \in \mathrm{Pic}^0(A)$, $S_-^0(X, K_X + D + D_1 + D_2 + a^*\mathcal{P}_{\alpha_0})$ is birational.

(iv) Assume that $S_-^0(X_\eta, K_{X_\eta} + D_\eta)$ is birational, and D_1, D_2 are nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors such that $S_-^0(X_\eta, K_{X_\eta} + (D_i)_\eta) \neq 0$. Then for any $\mathcal{P}_{\alpha_0} \in \mathrm{Pic}^0(A)$, $S_-^0(X, K_X + D + (K_X + \ulcorner D_1 \urcorner) + (K_X + \ulcorner D_2 \urcorner) + a^*\mathcal{P}_{\alpha_0}))$ is birational.

Theorem 5.3. *Let X be a minimal terminal threefold of general type over an algebraically closed field of characteristic p.*

(1) Assume $q(X) > 0$. Then $S_-^0(X, K_X + nK_X) \neq 0$ if $n \geq 11$, and $S_-^0(X, K_X + nK_X)$ is birational if $n \geq 21$; and if moreover $p > 2$, then $S_-^0(X, K_X + nK_X) \neq 0$ if $n \geq 9$, and $S_-^0(X, K_X + nK_X)$ is birational if $n \geq 17$.

(2) Assume $q(X) = 0$ and X has only Gorenstein singularities. Set $n_0(2) = 13, n_0(3) = 10, n_0(5) = 9$, $n_0(p) = 8$ if $p \geq 7$. Then $S_-^0(X, K_X + nK_X) \neq 0$ if $n \geq 2n_0(p) + 2$ and $S_-^0(X, K_X + nK_X)$ is birational if $n \geq 3n_0(p) + 3$.

To apply RR formula, we need a Miyaoka-Yau type inequality.

- The classical Miyaoka-Yau inequality for minimal surfaces of general type in char 0, is
$$3c_2(X) - K_X^2 \geq 0.$$
- For minimal threefolds in char 0, Miyaoka proved that $c_2(X)$ is pseudo-effective, hence $K_X \cdot c_2(X) \geq 0$.
- In char p, Miyaoka-Yau inequality does not hold for surfaces, for minimal surfaces [GSZ19] proved
$$c_2(X) + \frac{5}{8}c_1(X)^2 \geq 0.$$
which is sharp!
- In dimension three, [Zha20b] proves
$$c_2(Z) \cdot \rho^* K_X + AK_X^3 \geq 0 \tag{1}$$
where $A = \frac{(54n_0^2+9n_0)p^2+(9n_0+\frac{3}{2})p}{(p-1)^2}$ and n_0 is the Cartier index of K_X.

5.4. **Problems.**

- It is expected that there is a constant $M(v)$ such that for any threefold X with volume $v(K_X) \geq v$ and any $n \geq M(v)$, $|nK_X|$ is birational.
- Seshadri constant: Is there a number $\epsilon(d)$ such that, for any smooth projective variety X of dimension d and any nef and big line bundle L on X, the Seshadri constant at very general point $x \in X$
$$\epsilon(L, x) = \mathrm{Sup}\{t|\mu^* L - tE \text{ is nef}\} \geq \epsilon(d).$$
This will give a lower bound $M(d)$ such that for $n \geq M(d)$, $K_X + nL$ is birational. In char 0, we may take $\epsilon(d) = \frac{1}{d}$ [EKL95].
- Effectivity for surfaces over non-algebraically closed field, which is needed if one wants to study effectivity of threefold by induction.
- Canonical bundle formula which will be discussed later.
- Miyaoka-Yau inequality for threefold.

6. Canonical bundle formula

The canonical bundle formula is developed to treat fibrations with relatively trivial (log-) canonical class.

6.1. **The formulation in characteristic zero.** Let (X, Δ) be a pair. A morphism $f : (X, \Delta) \to Y$ to a normal projective variety Y is a klt-trivial, respectively lc-trivial, fibration if:

(a) f is a surjective morphism with connected fibres,

(b) (X, Δ) has klt, respectively log canonical, singularities over the generic point of Y, (c) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $K_X + \Delta \sim_{\mathbb{Q}} f^*D$,

(d) there exists a log resolution $\mu : X' \to X$ of (X, Δ) such that, if E is the set of all geometric valuations over X which are defined by a prime divisor E on X' such that $a(E, X, \Delta) > -1$, and if we denote $\Xi' = \sum_E a(E, X, \Delta)E$ then rank $(f \circ \mu)_* \mathcal{O}_{X'}(\ulcorner \Xi' \urcorner) = 1$.

There is $\varphi \in K(X)$ such that $K_X + \Delta + \frac{1}{r}\mathrm{div}(\varphi) = D$. For a prime divisor P of Y, let $r_P = lct(X, \Delta, f^*P)$. Kawamata [Kaw98] defined the discriminant part $B_Y = \sum_P (1 - r_p)P$ and the moduli part M_Y by

$$K_X + \Delta + \frac{1}{r}\mathrm{div}(\varphi) = f^*(K_Y + B_Y + M_Y).$$

In fact B_Y, M_Y are b-divisors, that is, they are compatible with base changes.

By [Kaw98, Amb04, Amb05], for lc trivial fibration, there exists a birational morphism $Y' \to Y$ such that for any birational transformation $\nu : Y'' \to Y$ we have that $M_{Y''} = \nu^* M_{Y'}$ and $M_{Y'}$ is nef. Ambro [Amb04] also proved (Y, B_Y) is klt iff (X, Δ) is klt. **An important conjecture: M_Y is semiample.** This has been proved when fibers are elliptic curves, rational curves K3 surfaces or abelian varieties or if $M_Y \equiv 0$ ([Pro09] [Fuj03]). Remark that most known progresses are based on the explicit construction of moduli space.

The coefficient of the M_Y is of great significance to effectivity problems. [Flo14] proved that there exists $N(r, b)$ such that NM_Y is integral.

The main technical ingredients in the study of canonical bundle formula are variation of Hodge structure, and remark that the positivity of the moduli part also is a consequence of the application of VHS.

6.2. **The case in characteristic p.** In characteristic p, based on the moduli space of elliptic curves and marked pointed rational curves, we have similar canonical bundle formula as in characteristic 0. But as the geometric fibers can be singular (not even normal or reduced), it does not make sense to consider the discriminant part or the moduli part. In fact, even for quasi-elliptic fibration, it is easy to construct examples $f : X \to C$, a quasi-elliptic fibration from a surface to a curve, such that $\omega_{X/C} \sim_{\mathbb{Q}} -D$ for some effective divisor D on C.

Witaszek [Wit17] proves that: for a fibration $f : (X, \Delta) \to Y$ from a lc pair, which is fibred by rational curves and such that $K_X + \Delta \sim_{\mathbb{Q}} D$, assuming that $p > 3$ or $p > 2$ and Δ is big, there exists a purely inseparable morphism $\pi : Y' \to Y$, an effective divisor $\Delta_{Y'}$ on Y' and a rational number $t \in [0, 1]$ such that

$$\pi^* D \sim_{\mathbb{Q}} t\pi^* K_Y + (1 - t)(K_{Y'} + \Delta_{Y'}).$$

This formula follows from base change and adjunction formula. Remark that the above formula also holds for quasi-elliptic fibration by similar strategy.

In some cases, for example when Y is of m. A. d., this kind of formula implies nonvanishing results. But it is not enough for the study of effectivity problems.

Question 6. Can we have a refined canonical bundle formula for quasi-elliptic fibration, which can be applied to prove abundance for threefolds in lower characteristic and study the effectivity problem?

7. Varieties with $K_X \equiv 0$

7.1. **Results in characteristic zero.** Over the field of complex numbers $\mathbb{C}$, if X is a smooth projective variety X with $K_X \equiv 0$, then K_X is semiample, namely

$K_X \sim_{\mathbb{Q}} 0$, and there exists an étale cover $\hat{X} \to X$ such that

$$\hat{X} \cong \prod_v X_v \times T$$

where X_v is simply connected Calabi-Yau variety or hyperkähler variety, and A is an abelian variety. The strategy is explained as follows, first use Yaus solution to the Calabi conjecture in order to equip X with a Ricci-flat Kahler metric, and then apply the splitting theorem of Cheeger-Gromoll to split a finite étale cover of X according to its holonomy decomposition.

If X is a singular projective variety with klt singularities and numerically trivial canonical divisor, in a series work [GKP16, GGK19, HP19, Dru18] the authors proved the similar decomposition theorem, it is worth mentioning that the above decomposition also holds by admitting quasi-étale covers. Let me explain the techniques as follows.

[?GGK16]: for a canonical singularity X, (1) there exists a quasi-étale cover $A \times \tilde{X} \to X$, with A being an abelian variety, and $\hat{q}(\tilde{X}) = 0$, and (2) the tangent sheaf of $\tilde{X}$ splits into direct sum of integrable stongly stable sheaves with trivial determinants. Here

$$\hat{q}(X) := \max\{\dim Pic^0(X')|X' \to X \text{ is a quasi} - \text{etale cover}\}$$

is called augmented irregularity and strongly stable means the stability preserved by any quasi-étale cover $\pi : X' \to \tilde{X}$. (1) is essentially proved by [Kawamata85]. The decomposition (2) relies on recent extension results for differential forms on singular spaces [GKKP11] and the non-degeneracy of the paring [GKP16]

$$H^0(X, \Omega_X^{[p]}) \times H^0(X, \Omega_X^{[n-p]}).$$

This paper leaves the splitting problem with respect to strongly stable decomposition.

Then Druel refines the above decomposition when $\dim X \leq 5$, which tells that $\tilde{X} = \prod Y_i \prod Z_i$ where Y_i has $h^0(\Omega^q_{Y_i}) = 0$ for any $0 < q < \dim Y_i$ and Z_i is symplectic. And finally HP proved the decomposition for arbitrary dimensional varieties. The rough idea is first studying the splitting property of the tangent bundle, then proving the integrability of the factors. Finally we remark that the arguments are highly technical involving both deep results from differential geometry and algebraic method.

7.2. **Related results in characteristic p and further questions.** In positive characteristics, the structure of this kind of variety is still mysterious. Recently [PZ19] proved a decomposition result with additional assumption that X is globally F-split: there exists a quasi-étale cover $Z \to X$ such that Z is the quotient of $Y \times A$ by the group $\prod \mu_{p^i}$ where Y has $\hat{Y} = 0$ and A is an abelian variety.

7.3. **Further questions.** (1) For a threefold with $K_X \equiv 0$ and klt singularities, is K_X semiample, moreover is there an effective bound m such that $mK_X \sim 0$?

(2) For a variety with $K_X \equiv 0$ and klt singularities with $\hat{q}(X) = 0$, is the fundamental group finite?

(3) Without F-splitting assumptions, is there a similar decomposition structure?

8. Questions

The following is expected as we mentioned before.

Question 7. Assume $K_{X/Y} + \Delta$ is f-semiample and $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F-regular. Is $f_*\mathcal{O}_X(a(K_{X/Y} + \Delta))$ WP for sufficiently divisible?

By Patakfalvi's result Theorem 3.7, when $\dim Y = 1$, this is known to be true in two extremal cases where $K_{X/Y} + \Delta$ is f-ample or is f-trivial.

It is of special significance to study Iitaka conjecture for fibrations induced from Albanese maps.

Question 8. Let $a_X : X \to A$ be the Albanese map. Assume that $\kappa(X) = 0$. Is a_X a surjective map?

This was proved by Kawamata [Kaw81] in char 0, which plays an important role in the study of Albanese maps. In char p, if a_X is generically finite then a_X is a birational map [?HPZ17]. If fibers of a_X have higher dimension, we reduce to study Iitaka conjecture.

9. Irregular varieties

9.1. Generic vanishing.

9.2. Characterization of abelian variety.

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